Canonical Detection in Spherically Invariant Noise

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Abstract— **The paper deals with the detection of signals with unknown parameters in impulsive noise, modeled as a spherically symmetric random process. The proposed model subsumes several interesting families of noise amplitude distributions: generalized Cauchy, generalized Laplace, generalized Gaussian, contaminated normal. It also allows handling of the case of correlated noise by a whitening approach. The generalized maximum likelihood decision strategy is adopted, resulting in a *canonical* detector, which is independent of the amplitude distribution of the noise. A general method for performance evaluation is outlined, and a comprehensive performance analysis is carried out for the case of M–ary equal–energy orthogonal signals under several distributional assumptions for the noise. The performance is contrasted with that of the maximum likelihood receiver for completely known signals, so as to assess the loss due to the a–priori uncertainty as to the signal parameters.**

I. Introduction

In most applications of statistical decision theory to detection of signals in additive noise, Gaussian noise is assumed since other distributional assumptions usually lead to mathematical difficulties. However, in many practical instances the measured probability density function (pdf) of the additive disturbance exhibit much heavier tails than the Gaussian distribution. A number of models have been proposed for such an *impulsive* noise, either fitting experimental data or based on physical grounds. For a review of the most credited empirical and theoretical models see [1] and references thereof and [2, pp. 72–94].

Designing optimum detectors for these noise models requires the complete statistical specification of the received signal under each hypothesis. For the case that the useful signal is completely known, this amounts to specifying the noise process only. If noise samples are independent, a first order characterization suffices. If, more realistically, noise samples are correlated, a model is needed which completely specifies the noise starting upon a partial knowledge of the relevant statistics. In practice, however, the useful signal can hardly be considered completely known: then, the specification of the received signal entails a further analytical burden due to unknown or fluctuating parameters in the useful signal.

For the case of independent non Gaussian noise samples, the theory of *Locally Optimum Bayes Detection* (LOBD) can be applied: the likelihood functions are replaced by their Mc Laurin expansions, truncated to the first nonzero term, with possibly an additional bias term to ensure asymptotic optimality [1] The resulting decision structure depends upon the signal parameters, whether unknown or fluctuating, as well as on the noise pdf [3] It achieves *locally* (i.e., for vanishingly small signal-to-noise ratio (SNR)) minimum error probability, but performance may strongly depart from the theoretically optimum achievable one as the SNR increases [1,4]. Moreover, the extension to correlated observations turns out to be unwieldy and usually requires a suboptimal approach due to the incomplete knowledge of the noise statistics [5].

An alternative approach relies on removing the a-priori uncertainty about the signal parameters by performing their Maximum Likelihood (ML) estimates under each hypothesis. Decision is made according to a generalized Maximum A-posteriori Probability (MAP) rule, wherein the unknown parameters are replaced by their ML estimates. This approach is usually referred to as Generalized Maximum Likelihood (GML) rule [6, chap. VII].

In this paper we apply this approach to detection of M equally likely signals, with unknown amplitude and phase, in additive noise modeled as a Spherically Invariant Random Process (SIRP). Such a model is compatible, at least in the first-order pdf, with all of the most credited distributions for non–Gaussian, or impulsive, noise, and leads to a *canonical* receiver, namely one whose structure and operation are independent of the distribution of the noise. A general procedure for the assessment of the performance is outlined and examples referring to several non-Gaussian noise distributions are presented. The rationale for this comparative analysis is to evaluate to what extent the marginal pdf of the non-Gaussian noise affects the receiver performance, once the SIRP model is in force.

II. SIRP model for non–Gaussian noise

We deal with the M-ary hypothesis testing problem [6, chap. VII], [7, chap. XXIII]

$$H_i \ : r(t) = \alpha p_i(t) + c(t), \qquad i = 1, \dots, M \qquad (1)$$

where $p_i(t)$, $i = 1, \dots, M$ and $r(t)$ are the complex envelopes of the transmitted waveforms and of the received signal respectively, $c(t)$ is a complex (possibly correlated) non-Gaussian process, modeling the noise and $\alpha = Ae^{j\theta}$ is a (possibly unknown) complex parameter, accounting for the incomplete knowledge of the useful signal.

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Table I
Families of admissible densities

	$f_X(x)$	$f(s)$
Contaminated normal	$\sum_i \frac{\epsilon_i}{\sqrt{2\pi s_i^2\sigma^2}} \exp\left(-\frac{x^2}{2s_i^2\sigma^2}\right)$	$\sum_i \epsilon_i \delta(s-s_i)$
	Middleton Class-A $\epsilon_i = e^{-\nu}\frac{\nu^i}{i!}$; $s_i^2 = \frac{i/\nu+\lambda}{1+\lambda}$	
Generalized Laplace	$\frac{a}{\sqrt{\pi}\Gamma(\nu)}\left(\frac{a\|x\|}{2}\right)^{\nu-1/2} K_{\nu-\frac{1}{2}}(a\|x\|)$	$\frac{2\nu^\nu s^{2\nu-1}}{\Gamma(\nu)} e^{-\nu s^2}$
	$a>0$; $\nu>0$; $a^2=\frac{\sigma^2}{2\nu}$	
Generalized Cauchy	$\frac{a^{2\nu}\Gamma(\nu+1/2)}{\sqrt{\pi}\Gamma(\nu)}(a^2+x^2)^{-\nu-1/2}$	$\frac{2\nu^\nu}{s^{2\nu+1}\Gamma(\nu)} e^{-\nu/s^2}$
	$a>0$; $\nu>1$; $a^2=2\sigma^2(\nu-1)$	
Generalized Gaussian	$\frac{\nu}{2a\Gamma(1/\nu)} e^{[-(\|x\|/a)^\nu]}$	$\int_0^\infty \frac{\nu\cos\left(t^{\nu/2}\sin(\nu\pi/4)-\frac{t}{2s^2}\frac{\Gamma(1/\nu)}{\Gamma(3/\nu)}\right)}{s^2 e^{-t^{\nu/2}\cos(\nu\pi/4)}[2\pi\Gamma(1/\nu)\Gamma(3/\nu)]^{1/2}} dt$
	$a>0$; $0<\nu\le 2$; $a^2=2\sigma^2\frac{\Gamma(1/\nu)}{\Gamma(3/\nu)}$	

The noise model should match experimental data which usually consist only of the marginal pdf and the autocorrelation function (acf) of disturbance. This leaves some indeterminateness as many noise models may comply with these constraints. Further desirable requisites for a noise model are that it be physically consistent and mathematically tractable. The most credited physical model has been proposed by Middleton [8]: unfortunately, the higher-order characterization of such a process in the general case of arbitrary correlation turns out to be unwieldy. Another important model is the *exogenous* one which regards the noise as the product of a real, non-negative, wide-sense stationary process, $s(t)$, say, times a Gaussian (possibly complex) one, $g(t)$, say, independent of $s(t)$, namely $c(t) = s(t)g(t)$. It has been shown *physically consistent* with some important disturbance phenomena, such as atmospheric noise [9]. This model describes quite faithfully disturbance phenomena arising from doubly stochastic mechanisms, wherein a slowly varying component - sometimes referred to as *regime* process [9] - modulates a Gaussian component (accounting for the local validity of the Central Limit Theorem) with much shorter decorrelation time. If the observation time is short with respect to the coherence time of the modulating process, then $s(t)$ can be approximated by a random constant s, to be called the auxiliary variate, and the exogenous process degenerates into a SIRP, namely [10,11]

$$c(t) = sg(t) \ . \tag{2}$$

Thus a SIRP is essentially a conditionally Gaussian random process. The process $c(t)$ is real or complex according to $g(t)$: in the present paper we consider complex SIRP's since we deal with complex envelopes. Accordingly to (2) the mean and the acf of $g(t)$ coincide, except for scale factors, with those of $c(t)$. We assume that s has unit mean square value so that both the SIRP and the underlying Gaussian process share the same autocorrelation (such a normalization is possible whenever s possesses finite mean square value) but we do not assume any particular structure of the autocorrelation matrix of the complex process. We refer to Appendix A of [12] for possible structural constraints on such matrix deriving, e.g., from an assumption of stationarity. An important feature of SIRP's is that they are completely specified by the autocorrelation of the overall process and the *auxiliary* pdf $f(s)$. Precisely, let $\mathbf{c} = \mathbf{c}_I + j\mathbf{c}_Q$ be the complex row vector of N samples drawn from the process $c(t)$ which we assume zero–mean, without loss of generality: its statistical characterization amounts to assigning the joint pdf of the $2N$ components of the real vector $(\mathbf{c}_I, \mathbf{c}_Q)$. Denoting by $\mathbf{M} = E[(\mathbf{c}_I, \mathbf{c}_Q)^T(\mathbf{c}_I, \mathbf{c}_Q)]$ its covariance matrix, it follows from (2) that the multivariate pdf of a complex SIRP admits the expression

$$f_{\mathbf{c}}(\mathbf{x}) = f_{\mathbf{c}_I,\mathbf{c}_Q}(\mathbf{x}_I, \mathbf{x}_Q) = \frac{h_{2N}(\| \mathbf{x} \|_{\mathbf{M}})}{(2\pi)^N |\mathbf{M}|^{1/2}} \ , \tag{3}$$

where $\mathbf{x} = (\mathbf{x}_I, \mathbf{x}_Q)$ is the $2N$-dimensional real argument, $\| \mathbf{x} \|_{\mathbf{M}} = \sqrt{\mathbf{x}^T\mathbf{M}^{-1}\mathbf{x}}$ is the norm induced by the positive definite matrix $\mathbf{M}^{-1}$ and $h_{2N}(x)$, $x \geq 0$, is the decreasing function

$$h_{2N}(x) = \int_0^{+\infty} \frac{f(s)}{s^{2N}} \exp\left(-\frac{x^2}{2s^2}\right) ds . \quad (4)$$

It follows from (2) that SIRP's are closed under linear transformations. In fact, if $c(t)$ undergoes a linear transformation, due to the scale property of linear systems and the closure property of Gaussian processes, this implies that the multivariate pdf of the transformed process is still given by (3), where $\mathbf{M}$ is now the covariance matrix of the samples of the output process [10,11]. It can also be shown that the quadrature components of a complex SIRP process share the same marginal pdf, designated $f_X(x)$, which is expressible as

$$f_X(x) = \int_0^{+\infty} \frac{1}{s\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2s^2\sigma^2}\right) f(s)\, ds , \quad (5)$$

where σ^2 is the variance of $g(t)$. Thus, (5) is an *admissibility* condition.

Some admissible densities are reported in Table I, where $\Gamma(\cdot)$ is the Eulerian Gamma function, $K_\nu(\cdot)$ is the modified Bessel function of second kind and order ν, a is a *scale* parameter related to the common variance σ^2 of the quadrature components, and ν is a *shape* parameter ruling the rate of decay of the noise pdf. The contaminated normal is the only distribution corresponding to a discrete auxiliary variate and subsumes the Middleton Class-A pdf whose parameter λ represents the ratio of the power of the Gaussian component of the noise to that of the impulsive (Poisson) one. Notice that compatibility between the Middleton Class-A model and the SIRP model is limited to the marginal pdf only, as the two models lead to different higher-order characterization. All these families include the Gaussian pdf as a special case: precisely, the generalized Gaussian yields the Gaussian distribution for $\nu = 2$, while the others reduce to the Gaussian distribution in the case $\nu \longrightarrow \infty$.

III. Synthesis of the receiver

A. Known signals

The following derivation of the ML receiver for detection of M known signals is a straightforward extension of previous results [13]. It lays the groundwork for the subsequent new derivation regarding detection of unknown signals and serves as a reference for performance assessment.

Let us assume, at first, that the noise process is uncorrelated. This ensures that the components of the complex envelope $c(t)$, in an arbitrary orthonormal basis, say $\{\psi_i(t)\}$, $i = 1, \ldots$, are uncorrelated (not necessarily independent) complex random variables, with a common variance equal to $2\mathcal{N}_0$, the Power Spectral Density (PSD) of $c(t)$: thus, the real and the imaginary part of these components are sequences with the same autocorrelation and zero cross-correlation [12]. Moreover, by the cited closure property, these sequences are SIRP's with the same auxiliary pdf as $c(t)$. Therefore a basis can be chosen whose first L components, $L \leq M$, span the subspace $\mathcal{C}_L$ of the transmitted waveforms $p_i(t)$, $i = 1, \ldots, M$. In anticipation of a limiting procedure for $N \rightarrow \infty$ we approximate the waveforms of (1) by their projections along $\psi_i(t)$, $i = 1, \ldots, N$, $N \geq L$, reducing the test to its *vector* form

$$H_i \ : \mathbf{r} = \alpha \mathbf{p}_i + \mathbf{c} ; \qquad i = 1, \ldots, M. \quad (6)$$

Being α a known parameter, the relevant likelihood functions for this test are

$$f_{\mathbf{r}}(\mathbf{r}|H_i) = f_{\mathbf{c}}(\mathbf{r} - \alpha\mathbf{p}_i) = \frac{h_{2N}\left(\frac{1}{\sqrt{\mathcal{N}_0}} \parallel \mathbf{r} - \alpha\mathbf{p}_i \parallel\right)}{(2\pi\mathcal{N}_0)^N} \quad (7)$$

$$i = 1, 2, \ldots, M$$

where $\parallel \cdot \parallel$ denotes Euclidean norm. As $h_{2N}(\cdot)$ is a decreasing function (for non-negative arguments), whichever the pdf $f(s)$ (see eq.(3)), the ML rule for perfectly known signals reduces to the conventional one based on minimizing, with respect to the signal index i, the distance $\parallel \mathbf{r} - \alpha\mathbf{p}_i \parallel$ or, equivalently, on maximizing the quantity $\Re\{\mathbf{r} \cdot \alpha\mathbf{p}_i\} - \frac{1}{2} \parallel \alpha\mathbf{p}_i \parallel^2$, where $\Re\{\cdot\}$ denotes real part. Since all $\mathbf{p}_i$'s have at most the first L coordinates non-zero, then the noise components orthogonal to $\mathcal{C}_L$ are *irrelevant*. Therefore, the test statistic does not depend on N, provided $N \geq L$, and in fact it can be expressed as

$$\Re\left\{\int r(t)\alpha^* p_i^*(t)\, dt\right\} - \frac{1}{2}\int |\alpha p_i(t)|^2\, dt , \quad (8)$$

where the integral is over the observation interval. The receiver thus consists of M matched filters: the real part of each output is corrected by a signal-dependent bias and subsequently fed to a largest-of selection device.

Let us now remove the assumption of uncorrelated noise, as it is unrealistic for the impulsive noise [14]. Modeling the impulsive noise as a correlated SIRP, one can take advantage of the closure of SIRP's under linear transformations. In fact, by preprocessing the received signal through a linear filter which whitens the disturbance, the problem of detecting the signals $p_i(t)$, $i = 1, \ldots M$ in correlated noise reduces to that of detecting the filtered signals $q_i(t) = \mathcal{L}p_i(t)$, $\quad i = 1, \ldots, M$, in uncorrelated noise. Therefore, the ML receiver for correlated SIRP noise is the same as for uncorrelated noise, but includes a whitening filter as its first stage. The resulting structure is as depicted in Fig. 1. We stress here that this receiver is *canonical* in the sense that its structure and operation are independent of $f(s)$ and, hence, of the specific distributional assumption about the noise, provided the admissibility condition (5) is fulfilled.

B. Signals with unknown parameters

We now proceed to the case of unknown signals, precisely to the case that neither the amplitude A nor the phase θ of α are known. We deal at first, with uncorrelated noise. To handle such an incomplete knowledge, the unknown parameter α in (6) might be considered as a complex random variate with pdf $p(\alpha)$ and the average likelihood $\int f_{\mathbf{r}}(\mathbf{r}|\alpha, H_i)p(\alpha)\, d\alpha$ might be maximized. To avoid dependence of the resulting receiver structure upon $p(\alpha)$ the most adverse $p(\alpha)$ might be selected according to the mini-max strategy [12].

A more convenient approach is the Generalized Maximum Likelihood (GML) rule: α is modeled as an unknown parameter, rather than as a random variate yielding an M-ary composite hypothesis testing problem (provided that all transmitted signals have non-zero energy). The hypothesis H_i is then selected for which the *generalized* likelihood function: $f_{\mathbf{r}}(\mathbf{r}|\widehat{\alpha}, H_i)$ is maximum, where $\widehat{\alpha}$ is the ML estimate of α. Based on (7), implementing the GML rule amounts to maximizing over the parameter space and the index space the quantity $h_{2N}(\mathcal{N}_0^{-1/2} \parallel \mathbf{r}-\alpha\mathbf{p}_i \parallel)$, or, equivalently, to minimizing the argument of $h_{2N}(\cdot)$ with respect to α and i. In accordance to GML rule, $\min_\alpha \parallel \mathbf{r}-\alpha\mathbf{p}_i \parallel = \parallel \mathbf{r}-\widehat{\alpha}\mathbf{p}_i \parallel$, and therefore the GML rule is still a minimum distance rule, but the distances are between the received vector $\mathbf{r}$ and the *estimated* signal vectors $\widehat{\alpha}\mathbf{p}_i$, $i = 1, \ldots, M$. Next, it is readily seen that $\parallel \mathbf{r}-\alpha\mathbf{p}_i \parallel^2$ is minimum at $\widehat{\alpha} = (\mathbf{r}\cdot\mathbf{p}_i)/\parallel \mathbf{p}_i \parallel^2$, so that the GML detector maximizes, with respect to i, the quantity $|\mathbf{r}\cdot\mathbf{p}_i|/\parallel \mathbf{p}_i \parallel$. As the noise components orthogonal to $\mathcal{C}_L$ are irrelevant, this statistic does not depend on N, provided $N \geq L$, and can be expressed as

$$\left| \int r(t) \frac{p_i^*(t)}{\parallel p_i(t) \parallel}\, dt \right| \quad , \quad i = 1, \ldots, M \ , \tag{9}$$

where the integral is over the observation interval. Thus the GML receiver selects the largest output from M envelopes detectors cascaded to as many filters, matched to the normalized signal waveforms.

The assumption of uncorrelated disturbance can now be removed resorting to a whitening filter (Sec. 3A), leading to the GML receiver structure for correlated non-Gaussian noise shown in Fig. 1, still canonical in the previous sense.

In case of equal energy signals, the GML detector reduces to the conventional *incoherent* receiver, optimum under Gaussian disturbance. Each composite hypothesis is the subspace, spanned by $p_i(t)$, of the transmitted signal vector space. If two or more such subspaces coincide - as it is the case when the respective transmitted waveforms are proportional (in the sense that they differ only for a complex factor) - then the corresponding hypotheses are completely overlapping in the parameter space, due to the effect of channel gain α, and hence are not resolvable even if the additive noise $c(t)$ is absent. Conversely, hypotheses corresponding to orthogonal subspaces are mutually exclusive in the absence of additive noise, independently of the channel gain, and hence are perfectly resolvable. Consequently, one might expect that the set of orthogonal signals is optimal.

In the case that one of the transmitted signals is zero, which is a case of one simple hypothesis versus $M-1$ composite alternatives, the GML strategy is inapplicable. For example consider On-Off Keyed (OOK) signals: $p_1(t) = 0$, $p_2(t) = p(t)$. Then the GML test is

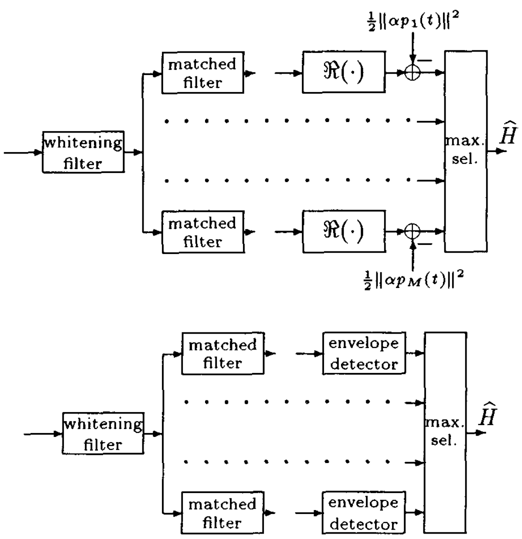

Figure 1: Schemes of the optimum ML (top) and GML (bottom) detector.

$$\text{if} \left| \int r(t)p^*(t)dt \right| > 0 \text{ then decide } H_1 \tag{10}$$

which is useless as the error probability is 1/2, whichever the SNR. To improve on this situation, one could resort to the so-called Generalized Likelihood Ratio (GLR) test. This can be considered as an extension of the GML obtained by allowing the threshold –which is unity for GML– to take on any value ≥ 1 so as to keep the type–I error to a prescribed level α while minimizing the type–II error. Accordingly, the total error probability decreases from 1/2 for no signal, to $\alpha/2$) for arbitrary large SNR. Unfortunately, implementing GLR requires a non–linearity depending on the noise distribution. Thus the receiver structure would be no longer canonical.

IV Performance assessment

The performance of the receiver for either known or unknown signals can be evaluated once $f(s)$ is given. In fact, on one hand receiver (9) coincides with the conventional incoherent receiver; on the other, any SIRP is a conditionally Gaussian process (see eqn.2). Thus, the error probability, $P(e)$, can be computed through

$$P(e) = \int_0^{+\infty} P(e|s)f(s)ds \ , \tag{11}$$

where $P(e|s)$ is the error probability of the receiver subject to Gaussian disturbance with zero mean and PSD $2s^2\mathcal{N}_0$. This shows that, just as in the case of Gaussian noise, receiver performance depends on the geometry of the adopted signaling scheme. Moreover, performance is expected to depend on the peak SNR's at the output of the matched filters, but to be otherwise independent of the transmitted waveforms $p_i(t)$.

M-ary orthogonal signaling

In principle, (11) allows one to evaluate the performance of any signal constellation for which the corresponding performance under Gaussian disturbance is known. We focus on the case of M orthogonal signals with equal energy $\mathcal{E}$. If the signals are completely known, use of a classical result of optimum detection in Gaussian noise [15, p. 152] yields

$$P(e|s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left\{1 - [1 - Q(x)]^{M-1}\right\} \exp\left[-\frac{1}{2}\left(x - \sqrt{2\gamma_R/s^2}\right)^2\right] dx \,, \tag{12}$$

where γ_R is the common SNR. For the case $M = 32$, the performance have been numerically computed for Middleton Class-A and Generalized Gaussian marginal pdf's of noise, and are reported as dashed curves (ML detector) in Figures 2 a,b respectively, representing $P(e)$ versus the per bit SNR $\gamma_b = \gamma_R/\log_2 M$. Here and in the sequel, we refer to the peak SNR at the output of the matched filter, which in turn is related to the input SNR γ_{in} through the so-called processing gain, G say, as

$$\gamma_{in} = \frac{A^2\mathcal{E}}{2\mathcal{N}_0} = \frac{\gamma_R}{G} \tag{13}$$

where $G = TB$ is the product of the signal bandwidth times the processing time. The quantity γ_{in} is meaningful in that, upon proper normalization, it represents the ***minimum detectable signal*** [7], namely the weakest signal that can be detected with prescribed $P(e)$. However, unlike the peak SNR γ_R, the minimum detectable signal depends upon the transmitted waveforms, which strongly affect the processing gain.

If the signals have unknown parameters, again from classical results of detection in Gaussian noise [15, p. 212], we have

$$P(e|s) = \sum_{k=1}^{M-1} \binom{M-1}{k} \frac{(-1)^{k-1}}{k+1} \exp\left(-\frac{\gamma_R}{s^2}\frac{k}{k+1}\right) . \tag{14}$$

Averaging with respect to the auxiliary pdf's under consideration yields, for Contaminated Normal noise:

$$P(e) = \sum_i \epsilon_i \sum_{k=0}^{M-1} \binom{M-1}{k} \frac{(-1)^k}{k+1} \exp\left\{-\frac{\gamma_R}{s_i^2}\frac{k}{k+1}\right\} \; ; \tag{15}$$

for Generalized Laplace noise:

$$P(e) = \frac{2}{\Gamma(\nu)} \sum_{k=1}^{M-1} \binom{M-1}{k} \frac{(-1)^{k-1}}{k+1} \left(\frac{k\nu\gamma_R}{k+1}\right)^{\nu/2} K_\nu\left(2\sqrt{\frac{k\nu\gamma_R}{k+1}}\right) \; ; \tag{16}$$

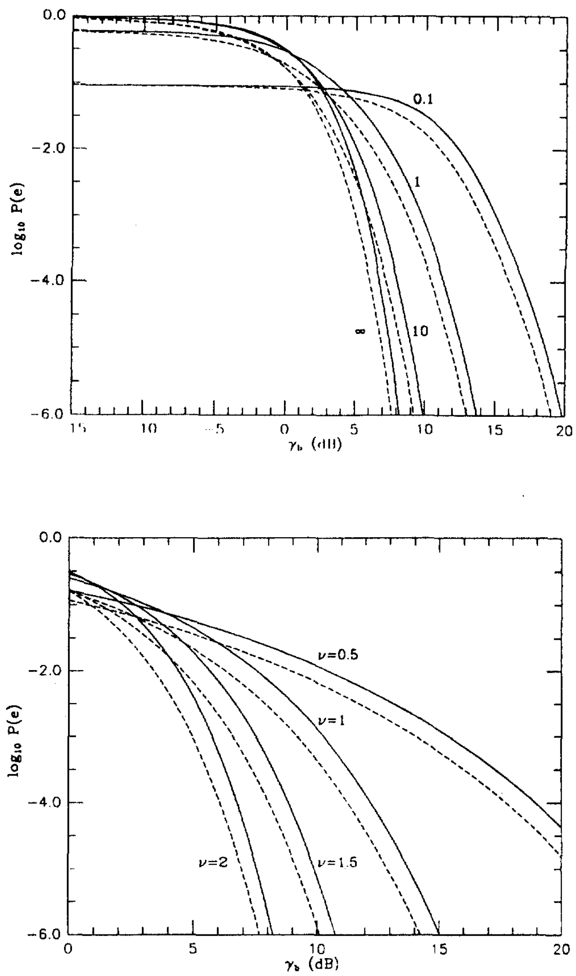

Fig. 2 ML (dashed) and GML (solid) receivers performance for M-ary orthogonal signals (M=32) with varying shape parameter: a) Middleton Class A marginal pdf; b) generalized Gaussian marginal pdf.

for Generalized Cauchy noise:

$$P(e) = \sum_{k=1}^{M-1} \binom{M-1}{k} \frac{(-1)^{k-1}}{k+1} \left[\frac{1}{1 + \frac{k}{k+1}\frac{\gamma_R}{\nu}}\right]^\nu . \tag{17}$$

In case of Generalized Gaussian noise, instead, a closed-form expression for $P(e)$ cannot be achieved, whence numerical integration techniques are to be adopted.

From equations (16) and (17), the limiting value of $P(e)$ for vanishingly small SNR is $1/M$, regardless the shape parameter of the noise pdf. The case of Middleton Class-A marginal pdf with no Gaussian component (i.e., $\lambda = 0$) is an exception to this trend. In this case the limiting value of $P(e)$, as evaluated from (15), is

$$\lim_{\gamma_R \to 0} P(e) = (1 - M^{-1})(1 - \epsilon_0) = (1 - M^{-1})(1 - e^{-\nu}) . \tag{18}$$

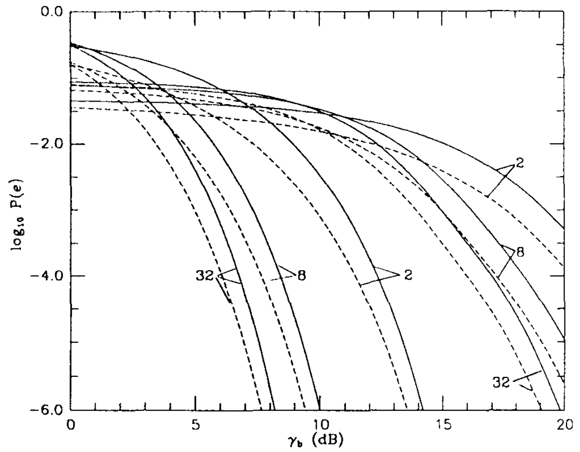

Fig. 3 GML receivers performance for the Middleton Class-A model. Orthogonal signaling with varying signal dimensionality and shape parameter (solid $\nu = .1$, dash $\nu = \infty$).

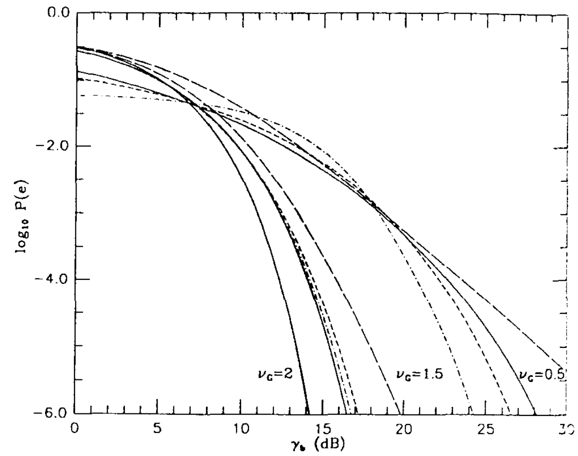

Fig. 4 GML receiver performance with varying shape parameter for: Generalized Gaussian noise (solid curve); Generalized Laplace noise (dashed curve); Generalized Cauchy noise (Long-dash curve); Middleton Class A noise (dot-dash curve).

This is not surprising, since for the case at hand there is a non-zero probability ($e^{-\nu}$) of no-noise in the observed signal. Correspondingly, for $\gamma_R \rightarrow 0$, $P(e)$ is zero with probability ϵ_0 and $1/M$ with probability $1-\epsilon_0$, which explains the dependence of the limiting value upon the shape parameter, as evidenced in Figure 2a (top). The performance for orthogonal, equi-energy signals with unknown parameters are shown versus γ_b as continuous plots in Figures 2 a,b, for $M = 32$ and for Middleton Class-A and Generalized Gaussian noise pdf, respectively. Comparison between corresponding characteristics for ML and GML detection allows one to evaluate the loss due to the partial knowledge about the signal parameters. This loss turns out to be influenced by the noise spikyness as well as by the SNR. As a general trend, spiky noise results in worse detection performance in the "strong signal" zone, and in enhanced detectability of "weak signals": also notice that the transition between the weak signal and the strong signal zone is smoother as noise spikyness increases. As expected, the curves exhibit a pedestal in the order of $1/M$, whatever the noise distribution and its shape parameter, with the relevant exception of noise with Middleton Class-A distribution with $\lambda = 0$, where the curves are attracted by the asymptote (18).

The effect of the size of the signal set can be elicited from Figure 3, where the performances for known and unknown signals are reported for Middleton Class-A noise with $\nu = 0.1$ and $\nu = \infty$ and for several values of M. In particular, the gain resulting from increasing the size M from 2 to 32 is quite uniform with ν: for example, at $P(e) = 10^{-6}$, it is in the order of 5 dB for both spiky ($\nu = 0.1$) and Gaussian noise ($\nu = \infty$).

Sensitivity analysis

The above performance analysis shows that, whatever the distributional assumption on the noise, the shape parameter is quite influential for detection performance. This, and the invariance of the receiver with respect to the noise distribution, suggest a sensitivity study aimed to ascertain whether the detection performance actually depends upon the parameter ν but is otherwise only marginally affected by the distribution itself. To this purpose, we evaluate detection performance under several instances of noise distribution, with the constraint that their respective shape parameters are set so as to achieve matching in the first two non-zero moments. We limit this analysis to the case of GML detection. Denoting by ν_A, ν_G, ν_C and ν_L the shape parameters of the Middleton Class A with $\lambda = 0$, Gaussian, Cauchy and Laplace generalized distribution, respectively, the values which ensure the quoted matching are the solutions of the following system of equations

$$\frac{\Gamma(1/\nu_G)\Gamma(5/\nu_G)}{\Gamma(3/\nu_G)^2} = \frac{3(\nu_C - 1)}{(\nu_C - 2)} \quad \nu_L = \nu_A = \nu_C - 2 . \quad (19)$$

Figure 4 reports performance curves of the binary orthogonal signaling scheme subject to Middleton Class A, generalized Gaussian, generalized Laplace and generalized Cauchy noise, respectively. All curves are labelled by the shape parameter ν_G although the actual parameters were selected according to (19). Obviously, for $\nu_G \rightarrow 2$ all of the curves approach the error probability curve under Gaussian noise (Gaussian limit). For lower values of ν_G the performance curves are still close to each other, except the one for Generalized Cauchy noise which significantly departs from the others as the SNR increases. A possible justification of the singularity of the Cauchy pdf is that it decays according to an algebraic law, while all of the others decay exponentially. From the results of Figure 4 it may be inferred that the shape parameter is the crucial factor in explaining the variation of performance and that additional factors tied to the particular noise distribution only account for minor residual variation, except possibly for factors tied to the type of asymptotic decay (algebraic or exponential).

V Conclusions

We have addressed a problem of receiver design for signals with possibly unknown parameters in non-Gaussian noise. The noise model based on the theory of SIRP's provides the mathematical tractability required to carry out the design, while embracing those marginal probability density functions most commonly assumed for impulsive disturbances. It also allows one to deal with correlated noise by a whitening approach. It is not fully compatible with Middleton impulsive noise model though, since the two models lead to different higher-order characterizations.

We adopted the ML decision rule, or its generalization, the GML rule, as needed in dealing with unknown signal parameters. In any case the test statistic turned out to be independent of the noise marginal pdf in the SIRP class, and coincident with the test statistic for Gaussian noise. In this sense, the resulting receiver for either completely known signals or for signals with unknown amplitude and phase parameters is canonical. In the special case of signals with equal-energy the resulting receiver is the conventional *coherent* or *incoherent* detector. Notice that the previous statements do not generally apply if the Maximum A Posteriori (MAP) decision rule is adopted.

We derived general formulas for the error probability, which also provide guidelines for the selection of the signal set. Generally speaking, optimum equi-energy signal sets under the assumption of Gaussian noise are also optimum under the assumption of any SIRP noise. In particular, the set of orthogonal equal energy signals is optimum for given size M, as it can be explained in the light of the composite hypotheses testing theory.

An investigation of a number of special cases showed that the noise distribution affects the performance mainly through the shape parameter, the scale parameter being accounted for by the SNR and other factors being relatively uninfluential. As a general rule, lower values of the shape parameter result in poorer performance.

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