Modelling and simulation of non-Rayleigh radar clutter

Prof. E. Conte, DrEng
Prof. M. Longo, DrEng
M. Lops, DrEng

Indexing terms: Radar, Clutter, Modelling, Simulation

Abstract: As the undesired echo from a radar environment (clutter) is often not Rayleigh-distributed, problems arise in specifying the clutter as a non-Gaussian random process and in computer-generating realisations of clutter for simulation purposes. The paper shows that modelling the clutter as a compound Gaussian or a spherically invariant random process allows a complete specification of the clutter suitable for use in radar design. It also lends itself readily to computer simulation procedures. Specific procedures for generating clutter realisations with assigned correlation properties and with K or Weibull amplitude distributions are suggested, and are validated by statistical tests. Then the issue of resolving between the Weibull and K distributions is considered, and empirical operative characteristics of the associated hypothesis test are computed.

1 Introduction

We consider the mathematical modelling of the undesired return from a radar environment (clutter). The value of an accurate model is twofold: at the analysis stage it allows the performance of radar processors to be predicted and at the design stage it allows optimum, or at least sub-optimum, receivers to be synthesised. It is also desirable, for cases where computer simulation is required, that clutter patterns conforming to the given model be easily generated by algorithmic procedures.

Clutter modelling involves in the first place the specification of a random process, that is the assignment of a family of distributions of finite dimensional random vectors that is *internally* consistent, in the sense that joint distributions of given order imply those of lower order. Such higher order joint distributions are of interest in the design of optimum radar processors, e.g. Neyman–Pearson detectors.

Further, in order that the model be realistic, some *external* consistency conditions must be satisfied to ensure that the expectations computed on the given model comply with those obtained from experimental data. Specifically, the expectations we have in mind are the amplitude distribution and the correlation functions.

Paper 7829F (E15), first received 14th March and in revised form 15th September 1990

The authors are with the Dipartimento di Ingegneria Elettronica, Università di Napoli, Via Claudio 21, 80125 Napoli, Italy

Regarding the amplitude distribution of the clutter, it is most often assumed to be a Rayleigh distribution, implying that the quadrature components of the received echo are jointly Gaussian processes: the usual intepretation is that the clutter arises from the superposition of returns from a large number of equivalent elementary scatterers, independent of one another, whence the resulting process is Gaussian as a consequence of the central limit theorem. However, experimental data indicate that large deviations from Rayleigh statistics are observed in situations such as low grazing angles and/or high-resolution radars. In such cases the empirical distributions exhibit both higher tails and larger standard deviation-to-mean ratio than predicted by the Rayleigh distribution (spiky clutter). Thus a single amplitude probability density function (APDF) cannot accommodate all of the different scenarios. Rather, we must consider a family of amplitude distributions, and we require that it include the Rayleigh one as a member.

This requirement excludes from further consideration the lognormal family which, although fitting data obtained in some experiments, does not include the Rayleigh distribution. Instead, the Weibull and the K distributions reduce to the Rayleigh distribution for special values of their parameters; moreover, both them have been validated by experimental data [1–8].

This paper introduces a coherent model for the clutter, i.e. one in which the clutter in each range bin is specified as a complex stochastic process, implying the joint statistical characterisation of arbitrary order of its quadrature components.

This model has to fulfil the cited constraints in terms of amplitude distribution and correlation functions.

We adopt a compound model for clutter, namely we assume that the received clutter echo is the product of a rapidly varying complex Gaussian process and an independent, slowly varying, modulating process; with this approach, the correlation properties of the overall process are dictated by those of the Gaussian component.

The compound model has been the subject of a substantial bulk of literature. In Reference 9, based on theoretical arguments, the PDF of the scattered power is modelled as a gamma variate with random parameters, but no attention is paid to the correlation properties of the received clutter echo.

With reference to K-distributed clutter, the compound model, in the form of the product of a Rayleigh rapidly varying process and a chi-distributed, slowly varying one, has been considered in References 3 and 5–7. Since the first experimental results, presented in Reference 3, the validity of this model for sea clutter has been supported by a complete set of evidence, as summarised in Reference 7, accounting for the effect of various geometrical and electromagnetic factors on the clutter statistics, such as the shape parameter of the K distribution and the correlation properties of both components.

The correlation properties of the slowly varying component are studied in Reference 10. Following that work, the extension to the bidimensional case is considered in References 11 and 12, and simulation algorithms for such slowly varying component are presented in Reference 13.

In this paper, a more general compound model is considered, in the sense that it allows in principle for a wider class of APDFs, although the worked examples refer to the Weibull and K distributions, since these are the APDFs for clutter returns which have received the most attention. We pay little attention to the correlation properties of the modulating component, since it is the overall correlation of the compound process which determines the system performance in the applications we have in mind, namely predicting and analysing the performance of radar processors subject to non-Gaussian clutter. In fact, in Reference 6 it has been shown that the spatial correlation of the clutter, which is reflected in the correlation of the slowly varying component, can always be taken into account, for performance prediction purposes, by suitably modifying the shape parameter of the received clutter.

For other applications, e.g. modelling the observed spatial properties of coherently illuminated surfaces [11], it is the slowly varying component that contains information about such properties. Thus, a different approach is required, in which the slowly varying component becomes the primary focus of the modelling and processing, whereas the rapidly varying component — called speckle in this content — is usually averaged out. In this sense, our work complements that presented in References 10 and 13.

2 Clutter modelling

2.1 Statement of the problem

We refer to the usual bandpass representation of the clutter:

$$x_{RF}(t) = \text{Re}\,[x(t)e^{j\omega t}] \tag{1}$$

where $x(t)$ is the complex envelope and can be written as

$$x(t) = x_c(t) + jx_s(t) = A(t)\exp\,[j\theta(t)] \tag{2}$$

in which $x_c(t)$ and $x_s(t)$ are the inphase and quadrature components, $A(t)$ is the envelope and $\theta(t)$ is the phase process. We consider the problem of specifying the complex envelope so that its first and second order statistics fulfil given constraints of consistency with physical requirements and of compliance with experimental results. With regard to the complex envelope of the clutter we impose two requirements:

(*a*) the APDF, namely the first-order PDF of $A(t)$, must belong to a prescribed family of distributions. This family should include the Rayleigh PDF as a member

(*b*) no constraint must be imposed on the correlation function of $x(t)$, i.e. the quantity

$$r_x(t, \tau) = E[x(t)x^*(t-\tau)] \tag{3}$$

can be arbitrarily assigned. Note that this *complex* correlation function amounts to the set of the following four real correlation functions:

$$\begin{aligned} r_{cc}(t, \tau) &= E[x_c(t)x_c(t-\tau)] \\ r_{ss}(t, \tau) &= E[x_s(t)x_s(t-\tau)] \\ r_{cs}(t, \tau) &= E[x_c(t)x_s(t-\tau)] \\ r_{sc}(t, \tau) &= E[x_s(t)x_c(t-\tau)] \end{aligned} \tag{4}$$

We also recall that specifying a process means assigning its joint PDFs of any order; therefore, a rule should be provided to derive such PDFs based on the correlation properties and on the APDF.

For sake of example, we focus on two families of ADPFs: the Weibull, namely

$$f_A(u) = acu^{c-1}\exp(-au^c) \quad u \geqslant 0 \tag{5}$$

where c is a shape parameter and a is related to the average power σ^2 of the quadrature components by

$$2\sigma^2 = a^{-2/c}\Gamma\left(1 + \frac{2}{c}\right) \tag{6}$$

and the K-APDF, namely

$$f_A(u) = \frac{b^{\nu+1}u^\nu}{2^{\nu-1}\Gamma(\nu)}K_{\nu-1}(bu) \quad u \geqslant 0 \tag{7}$$

where ν is the shape parameter, $\Gamma(\cdot)$ the Eulerian function and b is related to σ^2 by

$$b^2 = \frac{2\nu}{\sigma^2} \tag{8}$$

The Rayleigh APDF is the member of the Weibull family corresponding to the special case $c = 2$, as well as the member of the K-family corresponding to the limiting case $\nu = \infty$.

In these special cases the specification of the process is a problem which can be solved by classical methods since, as the quadrature components are jointly Gaussian processes, the joint PDFs of any order depend only upon the correlation function. One further advantage of Gaussian processes, related to simulation tasks, is their closure property under linear transformations. This property makes it possible to generate first a white Gaussian process and then to induce the desired correlation properties by linear filtering. We shall see shortly that these two features, namely the closure property and the specification of the process based solely upon the covariance matrix, are not unique to Gaussian processes but are shared by a broader class of processes called spherically invariant processes.

As the amplitude is no longer Rayleigh-distributed, the task of modelling and simulating radar clutter becomes more involved, since, in this case, assigning the correlation properties does not generally lead to the complete statistical characterisation of the process.

2.2 Generalised Wiener model

The Wiener model [17] for real processes with assigned first-order PDF and correlation function can be generalised to account for complex processes. The resulting scheme is outlined in Fig. 1, in which, as in the following

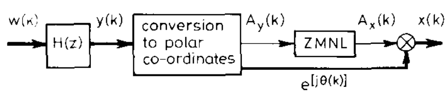

Fig. 1 *Model for a complex non-Gaussian correlated sequence: the Wiener approach*

figures, it is understood that the flow of complex quantities is indicated by thick lines. The desired process $x(k)$ is generated by transforming the envelope of a complex correlated Gaussian sequence via a zero memory nonlinearity (ZMNL), which leaves the phase process unchanged. The ZMNL converts the Rayleigh input APDF into the desired non-Rayleigh output APDF. The linear time-invariant filter $H(z)$ is included to induce the desired correlation properties in the final output $x(n)$. To this end, the relationship between the autocorrelation function of $y(k)$ and that of $x(k)$ across the nonlinearity, say

$$r_x(m) = g[r_y(m)] \tag{9}$$

should be determined, so that $H(z)$ can be calculated via spectral factorisation based on the autocovariance $r_y(m)$ resulting from eqn. 9. To be meaningful and realisable, this scheme requires that $r_y(m)$ be a covariance function, and in particular that it be non-negative definite. By referring to results of Reference 17 concerning the effects of nonlinear processing on the spectral (or correlation) properties of real (wide sense) stationary random processes, we observe that the above requirement is not always satisfied. For example, it cannot be satisfied — unless the ZMNL is a polynomial — if the spectrum of the output amplitude process $A_x(k)$ is to have some finite zeros, and in particular if it is strictly bandlimited. As a consequence, the autocovarance of $A_x(k)$, and hence the complex autocovariance $r_x(m)$, cannot be arbitrarily assigned for any shape of the ZMNL, that is to say for any desired output PDF.

This approach was first proposed in Reference 14 for lognormal clutter and then applied in References 15 and 16 to Weibull clutter. It is unmanageable, though, if the output envelope is to have the K distribution: in fact, an explicit expression for the ZMNL does not exist and, hence, no relationship between the input and the output correlation functions can be established. Therefore, we have to resort to a different approach, as outlined in the next subsection.

2.3 Exogenous model

With this model, introduced in Reference 18, the complex envelope $x(k)$ is considered to be the product $s(k)y(k)$ (Fig. 2), where $y(k)$ is a zero-mean complex Gaussian sequence,

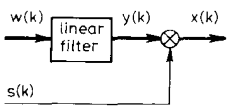

Fig. 2 *Exogenous model for a complex non-Gaussian correlated sequence*

and $s(k)$ is a real, non-negative, stationary sequence, independent of $y(k)$, which decorrelates on a time scale much longer than $y(k)$. The complex Gaussian process $y(k)$ of Fig. 2 may be modelled as a complex white Gaussian process $w(k)$ filtered through a linear (possibly time-varying) system, whose impulse response is determined by the correlation properties of $y(k)$. In other words, the clutter is modelled as a correlated Gaussian process modulated by the exogenous sequence $s(k)$.

The main feature of this model is that it allows independent control of the APDF, which is dictated by the first-order PDF $f(s)$ of the modulating sequence $s(k)$, and of the correlation properties, which are essentially those of the Gaussian process. In fact, by the total probability law, the envelope $A(k) = s(k)|y(k)|$ of the resulting sequence has the marginal PDF

$$f_A(u; k) = \int_0^\infty \frac{u}{s^2\sigma^2(k)} \exp\left(-\frac{u^2}{2\sigma^2(k)s^2}\right) f(s)\, ds \tag{10}$$

where $\sigma^2(k)$ is the mean square value of the clutter quadrature components and $s(k)$ has been assumed, without loss of generality, to have unit root mean square value. Thus, the amplitude distribution is uniquely determined by the marginal PDF $f(s)$ of the modulating process $s(k)$. Moreover, as $s(k)$ and $y(k)$ are independent, the complex correlation function of $x(k)$ is

$$r_x(n, m) = r_s(m) r_y(n, m) \tag{11}$$

where $r_s(m)$ is the autocorrelation of $s(k)$ and $r_y(n, m)$ is the (complex) autocorrelation of $y(k)$. Since $r_s(m) \simeq 1$ for any m where $r_y(n, m)$ is not vanishingly small, the overall correlation properties of $x(k)$ are practically those of the underlying Gaussian sequence.

Note that not all amplitude distributions admit the exogenous representation. In fact, a necessary and sufficient condition for a process $x(k)$ with APDF $f_A(u; k)$ to be admissible as an exogenous one is that there exists a PDF $f(s)$, $s \geqslant 0$, which satisfies the type-I integral equation of Fredholm (eqn. 10). It turns out that both the Weibull and the K distributions are admissible [19, 20]. The appropriate $f(s)$ for the K distribution is a generalised chi-PDF [5, 19, 21], namely

$$f(s) = \frac{2v^v}{\Gamma(v)} s^{2v-1} \exp(-vs^2) \quad s \geqslant 0 \tag{12}$$

The $f(s)$ for the Weibull distribution can be expressed in terms of Meijer's G functions, and hence reduces to some known function for special values of their parameters. However, in the general case an algorithm for its numerical computation is still lacking, so that such an expression is not very useful for practical purposes.

Besides the cited mathematical properties, exogenous models are appealing in that they can be physically interpreted in the light of the composite surface scattering theory [22]. According to this theory, the returns from any illuminated patch result from two independent contributions. The first component, with a Rayleigh-distributed envelope, accounts for the reflection from a very large number of independent, identically distributed elementary scatterers and exhibits a decorrelation period tied to the internal motion of the scatterers within the resolution cell as well as to the radar wavelength. A decorrelation time of the order of milliseconds is typical of sea clutter [3]. The second component accounts for the gross reflectivity characteristics of the illuminated patch and exhibits a decorrelation period which is associated with bunching of scatterers. For sea clutter, such a component decorrelates on a time scale of several seconds [3, 7]. This compound model is validated by the results reported in References 5 and 6, referring to sea clutter with K-distributed amplitude. The data reported therein indicate that any variation in radar resolution or grazing angle results in a corresponding variation of the shape parameter v of the generalised chi distribution of the modulating process $s(k)$. In Reference 6 it is also shown that the returns from groups of L correlated, identically distributed resolution cells can be related to a single return from a single, L times larger, resolution cell whose radar cross-section has a chi distribution whose parameter v depends upon the spatial correlation of the clutter.

2.4 SIRP model

The exogenous model allows the amplitude PDF and the correlation function to be controlled in accordance with eqns. 10 and 11, respectively, but otherwise does not specify the higher order statistics. To achieve the Nth order characterisation of the process (N finite) as required for the analysis of signal processors operating with blocks of size N, we resort to the SIRP model. A spherically invariant random process (SIRP) [23, 24] can be thought of as a 'degenerate' exogenous process in the limiting case of constant $s(k)$, in the sense that it is the product of a Gaussian process $y(k)$ and a *modulating* random variate s, rather than a modulating random process $s(k)$.

Two equivalent SIRP schemes are depicted in Fig. 3. Fig. 3*a* stems directly from the general block diagram of

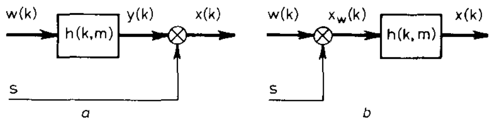

Fig. 3 *SIRP model for a complex non-Gaussian correlated sequence*
a Premodulation filtering
b Postmodulation filtering
$h(k, m)$ is the impulse response of the (possibly time varying) linear filter

an exogenous process (Fig. 2) with the sequence $s(k)$ replaced by the random variate s. The alternative implementation of Fig. 3*b* is derived from the scaling property of linear systems and is a direct proof of the *closure* of SIRPs with respect to linear transforms.

Most relevant features of exogenous processes apply to SIRPs as well: in particular, SIRP's accommodate both stationary and nonstationary situations, and allow the APDF and the correlation properties to be independently controlled. The applicability of the SIRP model for approximating the clutter process in sufficiently short time intervals is in keeping with the results in Reference 6.

The multivariate PDF of the complex SIRP $x(k)$ can be evaluated in a straightforward manner. From the scheme of Fig. 3*a* it is seen that, if we denote by

$$\boldsymbol{x} = \| x_{c_1}, \ldots, x_{c_N}, x_{s_1}, \ldots, x_{s_N} \|^T \tag{13}$$

a $2N$-dimensional vector whose entries are N samples from the inphase and quadrature components, then the PDF of $\boldsymbol{x}$ is

$$\begin{aligned} f_{\boldsymbol{x}}(\boldsymbol{x}) &= (2\pi)^{-N} |\boldsymbol{M}|^{-1/2} \int_0^{+\infty} s^{-2N} \\ &\quad \times \exp\left[-(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{M}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})/(2s^2)\right] f(s)\, ds \\ &= (2\pi)^{-N} |\boldsymbol{M}|^{-1/2} h_{2N}[(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{M}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})] \end{aligned} \tag{14}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{M}$ are the average and the covariance matrices of $\boldsymbol{x}$, respectively. Due to the structure of $\boldsymbol{x}$ (eqn. 13), its covariance matrix takes on the structure

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{cc} & \boldsymbol{M}_{cs} \\ \boldsymbol{M}_{sc} & \boldsymbol{M}_{ss} \end{bmatrix} \tag{15}$$

where $\boldsymbol{M}_{cc}$ and $\boldsymbol{M}_{ss}$ are the covariance matrices of the inphase and quadrature components, respectively, and $\boldsymbol{M}_{cs}$ and $\boldsymbol{M}_{sc}$ are their crosscovariance matrices.

Since $f_{\boldsymbol{x}}(\boldsymbol{x})$ depends on $\boldsymbol{x}$ through the quadratic form $(\boldsymbol{x}-\boldsymbol{\mu})^T \boldsymbol{M}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$, the vector is referred to as a spherically invariant random vector or SIRV. The function $h_{2N}(\cdot)$ is non-negative and is related to the APDF of the SIRP by

$$h_{2N}(y) = (-2)^{N-1} \frac{d^{N-1}}{dy^{N-1}} [\sigma y^{-1/2} p_A(\sigma y^{1/2})] \tag{16}$$

where σ^2 is the common mean power of the quadrature components of clutter [19].

A further relevant feature of SIRPs concerns their reprentation in hyperspherical co-ordinates. Let $\boldsymbol{x} = \| x_1, \ldots, x_{2N} \|^T$ be a $2N$-dimensional zero-mean white SIRV. Its hyperspherical co-ordinates $(R, \phi_1, \ldots, \phi_{2N-1})$ are defined as

$$\begin{aligned} x_1 &= R \prod_{j=1}^{2N-1} \sin \phi_j \\ x_k &= R \cos \phi_{2N+1-k} \prod_{j=1}^{2N-k} \sin \phi_j \quad 2 \leqslant k \leqslant 2N-1 \\ x_{2N} &= R \cos \phi_1 \\ 0 &\leqslant R < \infty \quad 0 \leqslant \phi_{2N-1} \leqslant 2\pi \\ &\qquad 0 \leqslant \phi_j \leqslant \pi \quad j \neq 2N-1 \end{aligned} \tag{17}$$

The angular co-ordinates $(\phi_1, \ldots, \phi_{2N-1})$ are independent of each other as well as of the radius R, and their marginal PDFs do not depend on the marginal PDF of the modulating variable s. On the contrary, the PDF of R_N is specified by the APDF of the complex SIRP, as it can be written as [25]

$$f_R(v) = \frac{v^{2N-1}}{2^{N-1}\Gamma(N)} h_{2N}(v^2) \tag{18}$$

We refer to Appendix 8 for further details on the cited properties.

Unlike a general exogenous process, a SIRP is nonergodic [19]. However, this is not a drawback as long as we deal with short sequences. In fact, we can still assume that the clutter process is an exogenous one, in the general sense specified above. Nevertheless, if it is observed and processed on time intervals much shorter than the average decorrelation time of the modulating sequence $s(k)$ (as is actually the case for most practical situations), then it can be considered as a SIRP to a reasonable degree of approximation (s approximately constant).

3 Short-time simulation of non-Rayleigh clutter

We distinguish the task of generating clutter samples by 'blocks', whose duration is short compared to the decorrelation time of the modulating process $s(k)$, from that of generating infinitely long streams of clutter samples, or, practically, streams much longer than the said decorrelation time. We call the first task 'short-time simulation' and the second task 'continuous simulation'.

In a short-time simulation by blocks of length N, we have to generate a complex N-dimensional vector, whose components are samples from the complex envelope of the clutter process, or, equivalently, a $2N$-dimensional real vector (eqn. 13), whose components are samples from the inphase and quadrature components of the clutter.

Two different approaches can be envisaged, depending on whether the auxiliary variable s, whose PDF is the solution of eqn. 10, can be generated or not.

3.1 Generation using the auxiliary variable: K-APDF

If an efficient algorithm producing the auxiliary variable s exists, then a generation procedure based on either of the models of Fig. 3 suggests itself, and can be implemented by either of the schemes of Fig. 4. Here the transformation matrix $\boldsymbol{G}$ is determined by the correlation matrix $\boldsymbol{M}$ by spectral factorisation. More precisely, letting $\boldsymbol{E}$ be the matrix of the normalised eigenvectors of $\boldsymbol{M}$ and $\boldsymbol{D}$ be the diagonal matrix of its eigenvalues, the transformation matrix is simply [26]

$$\boldsymbol{G} = \boldsymbol{E}\boldsymbol{D}^{1/2} \tag{19}$$

Short-time simulation of processes with K-distributed amplitude can be performed by this method since, in this case, the auxiliary variable s is related to a gamma variate with parameters (v, v) by

$$\gamma = s^2$$

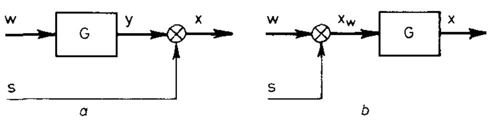

Fig. 4 *SIRP model for a complex non-Gaussian correlated sequence*
a Premodulation filtering
b Postmodulation filtering

and since efficient algorithms exist to produce a gamma variate. We cite the following from Reference 27:

(*a*) If v is an integer, then γ is the sum of v exponential variates with parameter v.

(*b*) If v is a semi-integer, then γ is a chi-square variate and hence can be easily generated as the square root of the sum of $2v$ squared independent standard Gaussian variates.

(*c*) If v is neither an integer nor semi-integer, it can be written as $v = v_1 + v_2$, with v_1 an integer or semi-integer and $0 < v_2 < 0.5$. Then γ is the sum $\gamma_1 + \gamma_2$ where γ_1 is gamma with parameter (v_1, v) and hence can be generated according to (*a*) or (*b*) above and γ_2 is gamma with parameters (v_2, v) and hence can be generated as the product

$$\gamma_2 = \eta\beta/v$$

where η is an exponential variate with unit parameter and β is a beta variate with parameters $(v_2, 1 - v_2)$. The beta variate β is in turn generated by the following standard rejection method [27]:

(i) generate two uniform variates with unit parameter, U_1 and U_2

(ii) evaluate $Y_1 = U_1^{1/v_2}$ and $Y_2 = U_2^{1/(1-v_2)}$

(iii) if $Y_1 + Y_2 > 1$ then go to step (i), otherwise deliver

$$\beta = \frac{Y_1}{Y_1 + Y_2}$$

Using the above procedure, we generated segments of length $N = 8$ from a zero-mean process with K-APDF, assuming the following correlation specifications:

$$\begin{aligned} \|\boldsymbol{M}_{cc}\|_{i,j} &= \|\boldsymbol{M}_{ss}\|_{i,j} = \rho^{(j-i)^2} \\ \|\boldsymbol{M}_{cs}\|_{i,j} &= \|\boldsymbol{M}_{cs}\|_{i,j} = 0 \end{aligned} \tag{20}$$

where ρ is the one-lag correlation coefficient. Owing to these specifications the complex envelope $x(t)$ is obviously wide-sense stationary (WSS) and its power spectral density (PSD) is Gaussian-shaped. Moreover, the radio-frequency process $x_{RF}(t)$ of eqn. 1 is itself WSS [28]. Fig. 5 compares the empirical cumulative distribution functions (CDFs) obtained from $M = 10000$ independently generated complex vectors, each of length $N = 8$, to corresponding theoretical CDFs, for several values of the shape parameter v, the second parameter b being determined from eqn. 8 so that $\sigma^2 = 1$ and hence $\|\boldsymbol{M}_{cc}\|_{i,i} = 1$. The goodness-of-fit was assessed through double Kolmogorov–Smirnov tests, using a population of size 10000 made of one complex component from each generated complex vector. The discrepancy indices were computed by subdividing this population into ten groups of size 1000; for all values of the shape parameter, the null hypothesis that the generated data were from the desired theoretical distribution could not be rejected at a significance level of 5%.

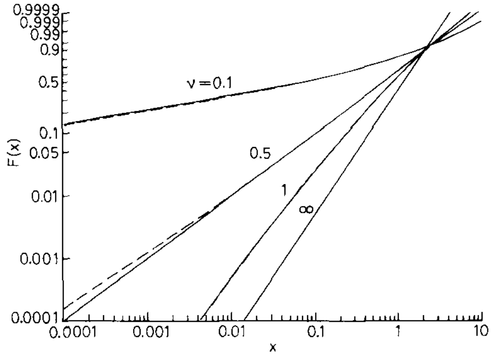

Fig. 5 *CDFs for K-APDF with varying shape parameter v (SIRP model)*
– – – – empirical
——— theoretical

With regard to the fulfilment of correlation requisites, we refer to Fig. 6, where the estimated entries of the covariance matrix $\boldsymbol{M}_{cc}$ for $v = 0.1$, namely the values $\hat{r}_{cc}(m)$, are shown and compared to the desired autocorrelation function, shown as a continuous line plot. Besides this visual appreciation, the goodness-of-fit of the estimated correlation to the theoretical one was tested via Fisher-z transformation of the calculated coefficients. Results indicate that the null hypothesis that the covariance matrix of the generated data is $\boldsymbol{M}_{cc}$ as given by eqns. 20 can never be rejected at a significance level of 5%. Similar results were obtained with reference to $\boldsymbol{M}_{ss}$.

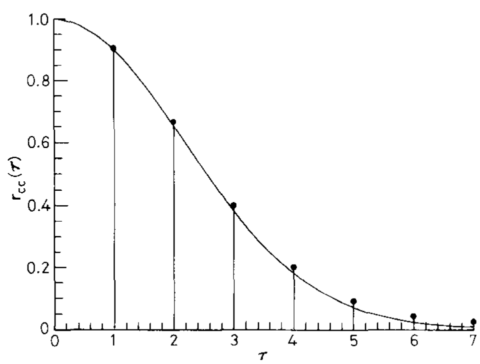

Fig. 6 *Estimated correlation values (discrete) and design correlation function (continuous) of correlated process with K-APDF (v = 0.1, SIRP model)*

3.2 Generation by co-ordinate transformation: the Weibull APDF

Lacking an algorithm for the generation of the modulating variate, we exploit the statistical properties of the hyperspherical co-ordinates of any SIRV cited in section

2 and suggest the generation scheme of Fig. 7. Starting with a white Gaussian random vector $\boldsymbol{w}$, the rectangular-to-spherical co-ordinates transformation produces a set of $2N - 1$ angular co-ordinates $\phi_1, \ldots, \phi_{2N-1}$, which are

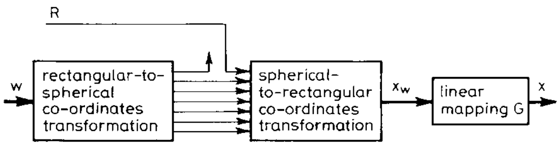

Fig. 7 *Generation scheme using spherical co-ordinates*

suitable for any SIRV, independently of the APDF. To produce the desired APDF, the radial co-ordinate R, separately generated according to this APDF, merges with the previous set of angular co-ordinates. Then a spherical-to-rectangular co-ordinate transformation yields the Cartesian representation of the SIRV $\boldsymbol{x}_w$ having the desired APDF but uncorrelated components. Finally, the desired covariance matrix is induced by the linear mapping $\boldsymbol{x} = \boldsymbol{G}\boldsymbol{x}_w$, with $\boldsymbol{G}$ as given in eqn. 19, and a non-zero mean may be achieved by adding a constant at the output.

The only nontrivial step in this procedure is determining the PDF of the radius R to match the APDF specification. This step involves the concatenation of eqns. 16 and 18. The following development refers to the Weibull APDF.

Assuming, without loss of generality, that the clutter quadrature components have unit mean square value, and using the Weibull APDF (eqn. 5) with parameters (a, c), eqn. 16 gives

$$h_{2N}(y) = (-2)^{N-1} \frac{a^{N-1}}{dy^{N-1}} [acy^{(c/2)-1} e^{-ay^{c/2}}] = (-2)^N \frac{d^N e^{-ay^{c/2}}}{dy^N} \tag{21}$$

Using the rule for the Nth derivative of a composite function [29], we obtain

$$\frac{d^N e^{-ay^{c/2}}}{dy^N} = \sum_{k=1}^{N} \frac{1}{k!} \left[\sum_{m=1}^{k} \binom{k}{m} (-1)^{k-m} x^{k-m} \frac{d^N x^m}{dy^N} \right] \times \left. \frac{d^k e^x}{dx^k} \right|_{x=-ay^{c/2}} = e^{-ay^{c/2}} \sum_{k=1}^{N} \frac{a^k}{k!} \times \left[\sum_{m=1}^{k} (-1)^m \binom{k}{m} y^{(c/2)(k-m)} \frac{d^N y^{m(c/2)}}{dy^N} \right] \tag{22}$$

Since

$$\frac{d^N y^{m(c/2)}}{dy^N} = \frac{\Gamma\left(m\frac{c}{2} + 1\right)}{\Gamma\left(m\frac{c}{2} + 1 - N\right)} y^{m(c/2)-N} \tag{23}$$

we can write

$$h_{2N}(y) = \sum_{k=1}^{N} A_k y^{k(c/2)-N} e^{-ay^{c/2}} \tag{24}$$

where

$$A_k = \sum_{m=1}^{k} (-1)^{m+N} \frac{a^k 2^N}{k!} \binom{k}{m} \frac{\Gamma\left(m\frac{c}{2} + 1\right)}{\Gamma\left(m\frac{c}{2} + 1 - N\right)} \tag{25}$$

Finally, using eqn. 18, we obtain the PDF of R:

$$f_R(v) = \frac{1}{2^{N-1}\Gamma(N)} \sum_{k=1}^{N} A_k v^{kc-1} e^{av^c} \tag{26}$$

For purposes of simulation, one is interested in the CDF $F_R(r)$. Integrating eqn. 26 yields the following expression:

$$F_R(v) = \sum_{k=1}^{N} C_k \left[1 - e^{-av^c} \sum_{m=0}^{k-1} \frac{(av^c)^m}{m!} \right] \tag{27}$$

with

$$C_k = \sum_{m=1}^{k} \frac{(-1)^{N+m}}{k(N-1)!} \binom{k}{m} \frac{2\Gamma\left(m\frac{c}{2} + 1\right)}{c\Gamma\left(m\frac{c}{2} + 1 - N\right)} \tag{28}$$

This procedure was implemented for generating segments of length $N = 8$ from a sequence with Weibull APDF, using the same correlation specifications as in the previous K-APDF example. Again, the empirical CDFs were

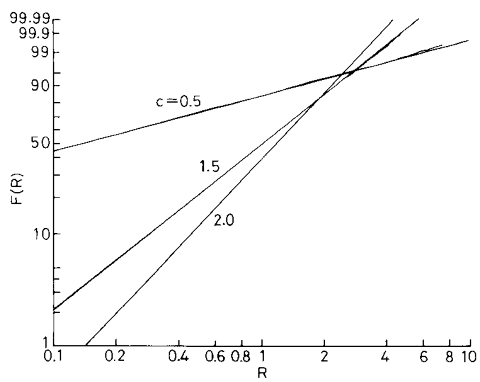

Fig. 8 *Empirical and theoretical CDFs for a Weibull ADPF with varying shape parameter c (SIRP model)*

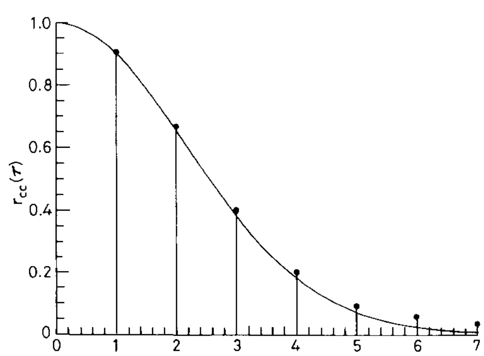

Fig. 9 *Estimated correlation values (discrete) and design correlation function (continuous) of correlated process with Weibull APDF*
$c = 1.5$, SIRP model

computed from $M = 10000$ realisations, using several values of the shape parameter c and selecting the second parameter a so that $\sigma^2 = 1$ from eqn. 6. These empirical CDFs were compared to the corresponding theoretical ones both graphically (Fig. 8) and using the double Kolmogorov–Smirnov test, with the same data partition of 10 groups of 1000 samples; again, the null hypothesis of Weibull APDF could not be rejected at a significance level of 5%.

The fulfilment of the correlation specifications was assessed by the same statistical procedures as outlined for the previous K-APDF example. Fig. 9 compares the computed $\hat{r}_{cc}(m)$ against the desired $r_{cc}(\tau)$, for $c = 1.5$. The Fisher-z test yielded acceptance of the null hypothesis of compliance at a significance level of 1%.

4 Continuous simulation

In this case the time fluctuations of the modulating process are no longer negligible with respect to the duration of the sequence to be generated, so that the SIRP model does not apply and a general exogenous model is needed. According to the scheme of Fig. 2, we are to generate a highly correlated sequence $s(k)$, namely one whose normalised correlation function is close to one as specified with reference to eqn. 11. Since the actual shape of the autocorrelation is immaterial, one possible generation scheme is that of Fig. 10.

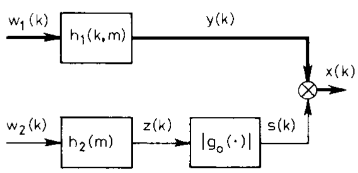

Fig. 10 *Generation scheme for a correlated exogenous sequence*
$h_1(k, m)$ is the impulse response of a possibly time-varying filter; $h_2(m)$ is the impulse response of a linear time-invariant filter

Here, a white complex Gaussian sequence $w_1(k)$ is generated and is fed to a linear filter whose time-lag impulse response $h_1(k, m)$ is specified by the desired overall correlation functions. If the process to be simulated is WSS, then this filter is linear time invariant and its energy transfer function $|H_1(\theta)|^2$ is proportional to the output PSD.

Independently, a white real Gaussian sequence $w_2(k)$ is generated and filtered by a linear time-invariant system with impulse response $h_2(m)$, and is passed to a ZMNL, $g_1(z)$ say, designed so that the output sequence has the desired marginal PDF $f(s)$. It is advantageous to implement $g_1(z)$ in the form

$$g_1(z) = |g_0(z)|$$

where $g_0(z)$ is an odd function of z, producing the symmetric PDF

$$f_{s0}(s) = \frac{f(s) + f(-s)}{2} \quad -\infty < s < \infty \tag{29}$$

so that $g_1(z)$ is given by

$$g_1(z) = |g_0(z)| = |F_{s0}^{-1}(1 - Q(z))| \tag{30}$$

Here $Q(x)$ is the area under the trailing tail of the standard Gaussian variate and $F_{s0}(\cdot)$ is related to the CDF $F_s(\cdot)$ of $s(k)$ as

$$F_{s0}(s) = \frac{1}{2} + \frac{F_s(s) - F_s(-s)}{2} \tag{31}$$

The autocorrelation of $s(k)$ depends on both $h_2(m)$ and $g_0(\cdot)$. Typically, the nonlinear processing enlarges the bandwidth, which does not meet the requirement that $r_s(m)$ should remain close to one. Therefore $h_2(m)$ should be designed such that the bandwidth prior to the nonlinear processing is sufficiently narrow for the cited requirement not to be endangered. Thus the requirements on $h_2(m)$ are essentially in terms of bandwidth, so that the classical design procedures for lowpass digital filters are adequate, with bandwidth specifications dictated by the criterion that the correlation of $s(k)$ should be close to unity in the range of interest for $r_x(m)$. In the case of nonstationary $x(k)$, a trial-and-error procedure might be necessary.

The procedure of Fig. 10 was applied for the generation of a WSS complex sequence with K-APDF and the following correlation specifications:

$$r_{cc}(m) = r_{ss}(m) = (0.9)^{|m|}$$
$$r_{cs}(m) = r_{sc}(m) = 0 \tag{32}$$

In this case the filter $h_1(m)$ is a simple autoregressive system of the first order:

$$h_1(m) = (1 - 0.81)^{1/2}(0.9)^m u(m)$$

where $u(m)$ is the (discrete) unit step function and the ZMNL is implicity defined by

$$\frac{\gamma(v, vs^2)}{\Gamma(v)} u(s) = 1 - Q(z)$$

where $\gamma(q, p)$ is the incomplete Eulerian function. In Fig. 11 plots of the function $g_0(z)$ are shown for increasing

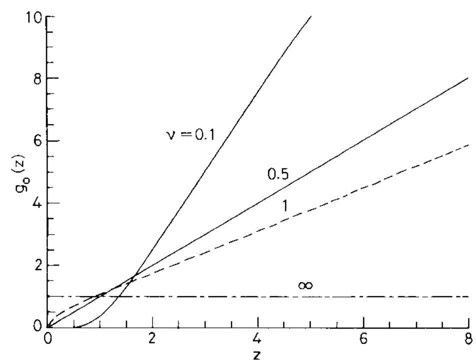

Fig. 11 *Behaviour of $g_0(z)$ for various shape parameters*
Only positive values are represented, since $g_0(-z) = -g_0(z)$

values of v. As v tends to infinity, $s(k)$ converges to a constant, which implies that $|g_0(z)| = 1$, $\forall z$. Note that the implemented scheme does not allow the shape of the autocorrelation of the modulating sequence to be easily predicted, but this is not a true drawback in our approach, in the sense specified in the introduction. However, if one is interested in controlling the autocorrelation of $s(k)$, several simulation techniques can be adopted. In particular, one can generate the process $s^2(k)$

by resorting to either of the two simulation algorithms introduced in Reference 13, which allow correlated observations to be generated from a gamma-distributed process. The first algorithm achieves the desired PDF, but is applicable only for obtaining an exponential correlation; the second algorithm on the other hand, allows wider control of the correlation function, but achieves the wanted PDF only approximately. Even so, however, an exact control of the final correlation of the modulating process requires a knowledge of an analytical relationship between the correlation functions of $s^2(k)$ and $s(k)$.

For several values of v, a sequence of 10000 samples was generated and empirical CDFs were computed and compared with theoretical CDFs. The results are presented in Fig. 12.

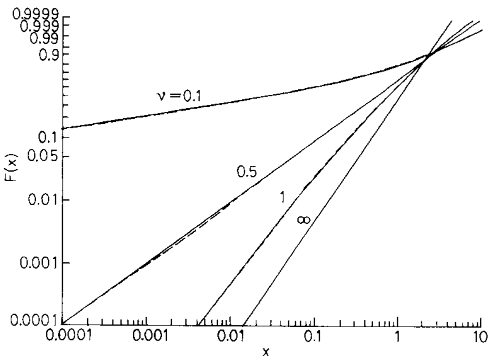

Fig. 12 *Empirical and theoretical CDFs for K-APDF with varying shape parameter v (exogenous model)*
- - - - empirical ——— theoretical

Since independent realisations were not available, an indirect test of goodness-of-fit was carried out exploiting the necessary and sufficient condition that the modulating variate s has the generalised-chi distribution of eqn. 12 in order that the exogenous sequence has the K-APDF. Therefore, we tested the hypothesis that the ZMNLs corresponding to the implemented v's convert a Gaussian variate into a generalised chi variate. After Kolmogorov–Smirnov tests, this null hypothesis could not be rejected at the significance level of 5%.

The fulfilment of the correlation requirements was tested on a generated K-distributed sequence with parameter $v = 0.5$. In Fig. 13 the estimated correlation values $\hat{r}_{cc}(m)$ are compared with the design correlation function $r_{cc}(\tau)$. After the Fisher test the null hypothesis that the generated sequence possesses the design correlation could not be rejected at a significance level of 5%.

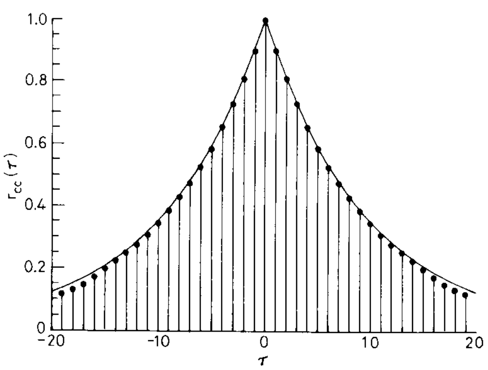

Fig. 13 *Estimated correlation values (discrete) and design correlation function (continuous) of correlated process with K-APDF*
$v = 0.5$, exogenous model

5 Resolvability of the Weibull and the K distributions

Several authors have referred to either Weibull or K distributions after studying their goodness-of-fit to the available experimental data [1, 6]. The question has then been raised as to whether the two distributions are essentially equivalent, whenever their parameters are connected by an appropriate relationship [30, 31].

A possible relationship results from the method of moments (MM) by forcing the first two moments of the two distributions to be equal, so that their parameters are implicitly defined by the nonlinear equations system

$$a^{-1/c}\Gamma\left(1 + \frac{1}{c}\right) = \frac{2\sqrt{(2)}}{b}\frac{\Gamma(v + \frac{1}{2})}{\Gamma(v)}$$
$$a^{-2/c}\Gamma\left(1 + \frac{2}{c}\right) = \frac{4v}{b^2} \qquad (33)$$

where a, c, b, v have been introduced in eqns. 5 and 7. Using this criterion, the closeness of the two distributions has been studied in Reference 31 and it has been shown that the theoretical CDFs exhibit similar behaviour.

An alternative relationship is based on a criterion of minimum distance (MD), in the Kolmogorov–Smirnov sense, between the Weibull and K distributions. If we denote by $F_W(x; c)$ the Weibull CDF with parameter c (the parameter a being such that $\sigma^2 = 1$ in eqn. 6) and by $F_K(x; v)$ the K-CDF with parameter v (the parameter b being such that $\sigma^2 = 1$ in eqn. 8), then their Kolmogorov–Smirnov distance is defined as

$$d(c, v) = \max |F_W(x; c) - F_K(x; v)| \quad x \geqslant 0 \qquad (34)$$

The relationship between the parameters c and v under the MM and MD criteria are plotted in Fig. 14; note that

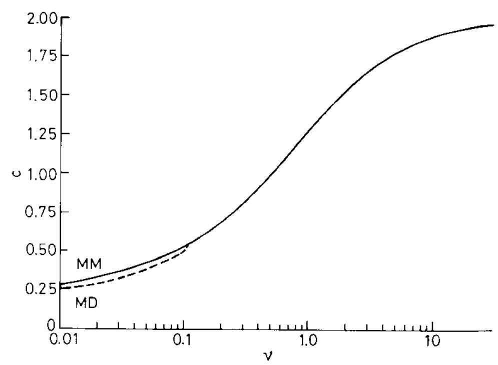

Fig. 14 *Relationship between the shape parameters of K and Weibull APDFs under method of the moments (MM) and minimum distance (MD) criteria*

the two criteria are equivalent except for very low values of v (or c), that is for very large deviations from the Rayleigh CDF.

The problem of discriminating between these two populations can be handled as a hypothesis testing problem. Thus, letting $\mathscr{X}_1$ be a sample of size M drawn from a Weibull population with CDF $F_W(x; c)$ and $\mathscr{X}_2$ be a sample of the same size from a K-population with CDF $F_K(x; v)$, we consider the decision problem

$$H_0: \mathscr{X}_1 \text{ and } \mathscr{X}_2 \text{ are drawn from the same population}$$
$$H_1: \mathscr{X}_1 \text{ and } \mathscr{X}_2 \text{ are } \textit{not} \text{ drawn from the same population} \quad (35)$$

As the test statistic we adopt the usual Kolmogorov–Smirnov measure of discrepancy [32], that is

$$D_M = \max |F_1(x) - F_2(x)| \quad x \geqslant 0 \quad (36)$$

wherein $F_1(\cdot)$ and $F_2(\cdot)$ are the empirical CDFs of $\mathscr{X}_1$ and $\mathscr{X}_2$, respectively.

The empirical operating characteristics of this test were evaluated for several values of the sample size M and for different values of the distributions parameters. The results are shown in Figs. 15 and 16, and demonstrate the power of the test (that is, $1 - \beta$, if β is the probability of a type-2 error) against the probability of a type-1 error (that is α, or significance level).

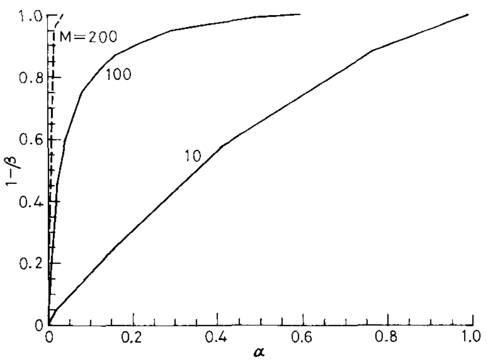

Fig. 15 *Operating characteristics of the test resolving between K- and Weibull APDF for v = 0.1, c = 0.53 and varying sample sizes*

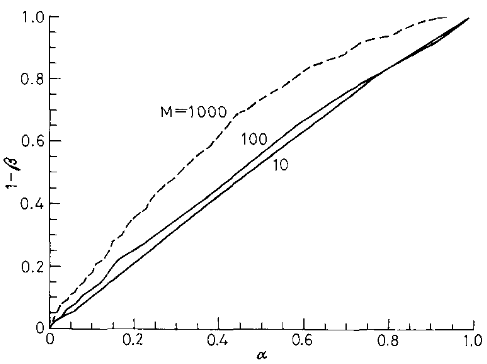

Fig. 16 *Operating characteristics of the test resolving between K- and Weibull APDF for v = 2, c = 1.53 and varying sample sizes*

With increasing sample size, the resolvability of the two distributions improves. However, the shape parameter is also influential. In fact, for $v = 0.1$ (hence $c = 0.53$ under the MM criterion), that is for high deviations from the Rayleigh CDF, the two distributions are resolvable with moderate sample sizes, such as $M = 100$ or 200. On the other hand, for $v = 2$ (hence $c = 1.53$ according to the MM criterion) the two distributions remain hardly resolvable, even if a sample size of $M = 1000$ is used.

6 Conclusions

This paper has addressed the problems of clutter modelling and simulation in situations where the Gaussian assumption no longer applies. The exogenous model can be adopted whenever the clutter amplitude is a compound-Rayleigh process, i.e. when it fulfils an *admissibility* condition (eqn. 10). From a physical viewpoint, this model corresponds to the composite surface scattering theory, which regards the clutter as the product of two independent processes: the spiky component, which controls the gross reflectivity characteristics of the illuminated patch and exhibits high average decorrelation time, and the speckle component, a Gaussian process which exhibits a much shorter decorrelation time.

The main advantage of exogenous models is that they allow the marginal PDFs of the clutter quadrature components and the covariance functions of the clutter complex envelope to be independently controlled. The former can be accommodated by a suitable choice of the marginal PDF of the modulating process, while the latter can be controlled by the correlation properties of the underlying Gaussian process. Moreover, if the clutter observation time is small enough, when compared against the average decorrelation time of the modulating process, then the spiky component can be thought of as a random variate, i.e. the overall process degenerates into a SIRP. In such a situation, the multivariate PDFs of any order can be written by a simple recursive formula once the covariance functions and the envelope PDF are known.

With regard to the simulation of non-Rayleigh clutter, some procedures are presented and implemented for generating clutter patterns with assigned envelope PDF and correlation properties. These techniques have been applied both to the case of Weibull-distributed clutter and to the case of K-distributed clutter. The results have been validated by statistical as well as graphical tests.

Applications of proposed modelling and simulation methods include the analysis and design of all radar processors involving a single observation and no adaptive thresholding technique. With some limitations, they also include cases where multiple observations are processed, either for adaptation purposes (CFAR) or for integration. Besides the trivial, though unrealistic case of independent observations, the case of a highly correlated modulating component of the clutter is fully amenable to the methods discussed here.

On the other hand, if the modulating component of the clutter is only partially correlated over the relevant observations (due either to spatial nonstationarity over the reference window in CFAR processing or to the dwell time exceeding the scale of stationarity of the gross reflectivity characteristics of the illuminated cell in the case of pulse integration) then the exogenous model is still applicable as long as the interest is in reproducing the correct autocovariance function and the correct APDF of the compound process, but it does not allow complete specification of the same compound process, i.e. knowledge of PDFs of any order.

A rather different approach is needed in applications, such as remote sensing, where the interest is not in the overall compound process, but only in its modulating component, since it contains information about the surface characteristics one is willing to sense.

An incidental result, which is made possible by the models proposed here, is the assessment of the extent to which the Weibull and K distributions are actually distinguishable from one another in statistical experiments. Our analysis indicates that the two distributions can be distinguished quite easily for very spiky clutter, in the sense that the Kolmogorov–Smirnov test has an acceptable resolving power for moderately low sample sizes. Conversely, if the clutter is not very spiky, the two distributions are practically indistinguishable, even if the sample size of the test is increased considerably.

7 References

1 SEKINE, S., OHATANI, S., MUSHA, T., IRABU, T., KIUCHI, E., and TOMITA, Y.: 'MTI processing and Weibull distributed clutter', *IEEE Trans.*, November 1982, **AES-18**, (4), pp. 729–730

2 SEKINE, S., MUSHA, T., TOMITA, Y., HAGISAWA, T., IRABU, T., and KIUCHI, E.: 'On Weibull distributed weather clutter', *IEEE Trans.*, 1979, **AES-15**, pp. 824–828

3 WARD, K.D.: 'Compound representation of high resolution sea clutter', *Elect. Lett.*, 6th August 1981, **17**, (16), pp. 561–563

4 WATTS, S., and WARD, K.D.: 'Radar sea clutter', *Microwave J.*, June 1985, **22**, pp. 109–121

5 WATTS, S.: 'Radar detection prediction in sea clutter using the compound K-distribution model', *IEE Proc. F, Commun., Radar & Signal Process.*, 1985, **132**, (7), pp. 613–620

6 WATTS, S., and WARD, K.D.: 'Spatial correlation in K-distributed sea clutter', *IEE Proc. F, Commun., Radar & Signal Process.*, 1987, **134**, (6), pp. 526–531

7 WARD, K.D., BAKER, C.J., and WATTS, S.: 'Maritime surveillance radar. Part 1: Radar scattering from the ocean surface', *IEE Proc. F, Radar & Signal Process.*, 1990, **137**, (2), pp. 51–62

8 WATTS, S., BAKER, C.J., and WARD, K.D.: 'Maritime surveillance radar. Part 2: Detection performance prediction in sea clutter', *IEE Proc. F, Radar & Signal Process.*, 1990, **137**, (2), pp. 63–72

9 LEWINSKI, D.J.: 'Nonstationary probabilistic target and clutter scattering models', *IEEE Trans.*, 1983, **AP-31**, (3), pp. 490–498

10 OLIVER, C.J.: 'Correlated K-distributed clutter models', *Opt. Acta*, 1985, **32**, (12), pp. 1515–1547

11 OLIVER, C.J.: 'The representation of correlated clutter textures in coherent images', *Inverse Prob.*, 1988, **4**, pp. 843–866

12 OLIVER, C.J.: 'The representation of radar sea clutter', *IEE Proc. F, Commun., Radar & Signal Process.*, 1988, **135**, (6), pp. 497–500

13 OLIVER, C.J., and TOUGH, R.J.A.: 'On the simulation of correlated K-distributed random clutter', *Opt. Acta*, 1986, **33**, (3), pp. 223–250

14 FARINA, A., RUSSO, A., and STUDER, F.A.: 'Coherent radar detection in lognormal clutter', *IEE Proc. F, Commun., Radar & Signal Process.*, 1986, **133**, (1), pp. 39–54

15 FARINA, A., RUSSO, A., SCANNAPIECO, F., and BARBAROSSA, S.: 'Theory of radar detection in coherent Weibull clutter', *IEE Proc. F, Commun., Radar & Signal Process.*, 1987, **134**, (2), pp. 174–190

16 LI, G., and YU, K.-B.: 'Modelling and simulation of coherent Weibull clutter', *IEE Proc. F, Radar & Signal Process.*, 1989, **136**, (1), pp. 2–12

17 WISE, G.L., TRAGANITIS, A.P., and THOMAS, J.B.: 'The effect of a memoryless nonlinearity on the spectrum of a random process', *IEEE Trans.*, 1977, **IT-23**, (1), pp. 84–89

18 CONTE, E., GALATI, G., and LONGO, M.: 'Exogenous modelling of non-Gaussian clutter', *J. Inst. Electron. & Radio Eng.*, 1987, **57**, (4), pp. 191–197

19 CONTE, E., and LONGO, M.: 'Characterisation of radar clutter as a spherically invariant random process', *IEE Proc. F, Commun., Radar & Signal Process.*, 1987, **134**, (2), pp. 191–197

20 CONTE, E., LOPS, M., and ULLO, S.: 'A new model for coherent Weibull clutter'. Proc. Int. Conf. on Radar, Paris, 24th–28th April 1989, pp. 482–487

21 JAKEMAN, E., and PUSEY, P.N.: 'A model for non-Rayleigh sea echo', *IEEE Trans.*, November 1976, **AP-24**, pp. 806–814

22 WRIGHT, J.W.: 'A new model for sea clutter', *IEEE Trans.*, March 1968, **AP-16**, pp. 217–223

23 BREHM, H.: 'Description and generation of spherically invariant speech-model signals', *Signal Process.*, 1987, **12**, (2), pp. 119–141

24 YAO, K.: 'A representation theorem and its application to spherically-invariant random processes', *IEEE Trans.*, September 1973, **IT-19**, (5), pp. 600–608

25 GOLDMAN, J.: 'Detection in the presence of spherically symmetric random vectors', *IEEE Trans.*, January 1976, **IT-22**, pp. 52–59

26 FUKUNAGA, K.: 'Introduction to statistical pattern recognition' (Wiley, New York, 1972)

27 RUBINSTEIN, R.: 'Simulation and the Monte-Carlo method' (Wiley, New York, 1981)

28 PAPOULIS, A.: 'Probability, random variables and stochastic processes' (McGraw Hill, 1965)

29 GRADSHTEYN, I.S., and RYZHIK, I.M.: 'Table of integrals, series and products' (Academic Press, London, 1983, 4th edn.)

30 D'ADDIO, E., GALATI, G., GIANNATEMPO, S., and PROTOPAPA, A.: 'Modelli non Gaussiani per il clutter superficiale', *Rivista Tecnica Selenia*, 1987, **10**, (3), pp. 23–38

31 WARD, K.D.: 'A radar sea clutter model and its application to performance assessment'. Proc. Radar-82 Conf., *IEE Conf. Publ. 216*, London, October 1982

32 MOOD, A.M., GRAYBILL, F.A., and BOES, D.C.: 'Introduction to the theory of statistics' (McGraw-Hill, 1974)

33 POLLARD, J.K.: 'A handbook of numerical and statistical techniques' (Cambridge University Press, 1979)

8 Appendix

Let $\boldsymbol{x}$ be a $2N$-dimensional white SIRV. By eqn. 14, its multivariate PDF can be written as

$$f_{\boldsymbol{x}}(\boldsymbol{x}) = \frac{1}{(2\pi)^N} h_{2N}(\boldsymbol{x}^T\boldsymbol{x}) \tag{37}$$

where $h_{2N}(\cdot)$ is given in eqn. 16. The Jacobian of the hyperspherical transformation eqn. 17 is

$$\left|\frac{\partial(x_1, \ldots, x_{2N})}{\partial(R, \phi_1, \ldots, \phi_{2N-1})}\right| = R^{2N-1} \prod_{j=1}^{2N-2} (\sin \phi_j)^{2N-1-j} \tag{38}$$

Therefore, the multivariate PDF of the hyperspherical co-ordinates of the vector $\boldsymbol{x}$ can be written as

$$f_{R, \phi_1, \ldots, \phi_{2N-1}}(v, \alpha_1, \ldots, \alpha_{2N-1}) = \frac{1}{(2\pi)^N} v^{2N-1} h_{2N}(v^2) \prod_{j=1}^{2N-2} (\sin \alpha_j)^{2N-1-j} \tag{39}$$

which proves that the hyperspherical co-ordinates are independent of each other.

Moreover, by writing the above expressions in a factorised form and introducing suitable normalisation factors, we find the following expressions for the marginal PDFs:

$$f_{R_N}(v) = \frac{v^{2N-1}}{2^{N-1}\Gamma(N)} h_{2N}(v^2) \quad 0 \leqslant v < +\infty$$

$$f_{\phi_{2N-1}}(\zeta) = \frac{1}{2\pi} \qquad 0 \leqslant \zeta \leqslant 2\pi$$

$$f_{\phi_k}(\zeta) = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\dfrac{2N-k+1}{2}\right)}{\Gamma\left(\dfrac{2N-k}{2}\right)} (\sin \zeta)^{2N-1-k} \qquad 0 \leqslant \zeta \leqslant \pi \quad k \neq 2N-1 \tag{40}$$