Sequential Detection of Markov Targets With Trajectory Estimation

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***Abstract*—The problem of detection and possible estimation of a signal generated by a dynamic system when a variable number of noisy measurements can be taken is here considered. Assuming a Markov evolution of the system (in particular, the pair signal–observation forms a hidden Markov model (HMM)), a sequential procedure is proposed, wherein the detection part is a sequential probability ratio test (SPRT) and the estimation part relies upon a maximum *a posteriori* probability (MAP) criterion, gated by the detection stage (the parameter to be estimated is the trajectory of the state evolution of the system itself). A thorough analysis of the asymptotic behavior of the test in this new scenario is given, and sufficient conditions for its asymptotic optimality are stated, i.e., for almost sure minimization of the stopping time and for (first-order) minimization of any moment of its distribution. An application to radar surveillance problems is also examined.**

***Index Terms*—Asymptotic optimality, hidden Markov models (HMM), sequential detection and estimation, sequential probability ratio test (SPRT).**

I. Introduction

MANY statistical decision problems in engineering applications require to perform state estimation of a dynamic system under uncertainty as to signal presence [1]–[3]. This includes fault detection and diagnosis in a dynamical system control [4], [5], target detection and tracking [6], image and speech segmentation [7], speaker identification, and source separation, blind deconvolution of communication channels. Application of sequential decision rules to the above scenario arouses much interest since it promises a considerable gain in sensitivity, measured by the reduction in the average sample number (ASN), with respect to fixed sample size (FSS) procedures. These advantages are particularly attractive in remote radar surveillance, where the signal amplitude is weak compared to the background noise and stringent detection specifications can be met only by processing multiple frames as in [8]–[10]). In this case, FSS techniques usually are inefficient while sequential procedures are known to increase the sensitivity of power-limited systems or, alternatively, to reduce the ASN.

The adoption of sequential procedures, however, poses some difficulties: since the instant when the procedure stops sampling is not determined in advance (it is a random stopping time, indeed) the set of trajectories of the dynamic system to be considered (i.e., the parameter space) has an infinite cardinality. On the other hand, sequential testing rules have been already extended to the case of composite hypotheses. In [11], a sequential probability ratio test (SPRT) is adopted in a radar framework assuming a prior on the parameter space, in turn consisting of a finite number of elements (the radar resolution cells). Suboptimal sequential classification procedures (also called multihypotheses tests) were also proposed during the past years, such as [12]–[19] for the case of independent and identically distributed (i.i.d.) observations and [20]–[24] for the more general setting of non-i.i.d. observations. However, all of these studies were restricted to a finite cardinality of the parameter space, an overly restrictive condition, which corresponds to requiring that the dynamic system may only lie in a determined state, with no transition allowed. Few works in the past have studied sequential problems for hidden Markov models (HMMs), which are known to admit a dynamical system representation in the sense of control theory [25]. In [26], the performances of SPRTs for model estimation in parametrized HMMs and the cumulative sum (CUSUM) procedure for change point detection in HMMs are studied, while [27] addresses the quickest detection of transient signals represented as HMMs using a CUSUM-like procedure, with possible applications to the radar framework.

This paper addresses the problem of sequential detection and trajectory estimation of the state evolution of a dynamical system observed through noisy measurements. In the above framework, its contributions can be summarized as follows.

- At the design stage, a sequential procedure is defined with no restriction as to the parameter space cardinality. The detection part of the procedure realizes an SPRT while, in order to estimate the system state trajectory, a gated estimator is defined, in the sense that estimation is enabled by the result of the detection operation.
- It is known that Wald's SPRT for testing simple hypotheses based on i.i.d. observations has a number of remarkable properties [28], [29], the most appealing being the fact that it simultaneously minimizes the expected sample size under both hypotheses. These properties, however, fail to hold when the observations are not i.i.d., as happens when they are generated by a dynamic system. In this paper, a deep asymptotic analysis for the detection part is given and sufficient conditions under which these properties hold are stated, consistent with previous results in [20], [21], [23]. In particular, it is shown that under a set of rather mild conditions the test ends with probability one and its stopping time is almost surely minimized in the class of tests with the same or smaller error probabilities. Furthermore, reinforcing one of such conditions, it is also shown that

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any moment of the stopping time distribution is first-order asymptotically minimized in the same class of tests.

— At the application stage, the general problem of multiframe target detection and tracking for radar surveillance is considered: in this way, previous limitations on target mobility imposed by other studies are avoided.

— Finally, a thorough performance analysis is given, aimed primarily at showing the correctness of the asymptotic analysis and at investigating the effects of system parameters. The superiority of sequential detection and estimation rules with respect to FSS techniques is also shown in the afore-mentioned radar application.

The rest of the paper is organized as follows. The next section presents the elements of the problem while Section III addresses the sequential detection and estimation problem. Section IV presents the asymptotic results while Section V covers the radar surveillance problem. Finally, Section VI is devoted to the presentation of numerical results, while concluding remarks are given in Section VII. For reader's sake, some notation, used throughout the rest of the paper, is first introduced.

Notation: In what follows, all random variables are defined on a common probability space $(\Omega, \mathfrak{F}, P)$ and are denoted with capital letters. Lower case letters are used to denote realizations of random variables while calligraphic letters to denote sets within which random variables take values. σ-algebras are denoted using script letters, $\sigma(X)$ being the smallest σ-algebra generated by the random variable X. $\boldsymbol{X}_{i:j}$ will be used to denote segments of random variables taken from the process $\{X_k\}_{k\in\mathbb{Z}}$: specifically, $\boldsymbol{X}_{i:j} = \{X_k\}_{k=i}^{j}$ for $i \le j$, and $\boldsymbol{X}_{-\infty:j} = \{X_k\}_{k=-\infty}^{j}$. E is the operator of expectation: a subscript will be added in case of ambiguity, so that E_θ and E_H are expectation when θ is the true state of nature and hypothesis H is true, respectively. $\mathrm{D}(\cdot\|\cdot)$ denotes the Kullback–Leibler divergence operator. The acronyms a.s. and a.e. stands for almost sure and almost everywhere. $\mathbb{N}$ denotes the set of natural numbers, i.e., $\{1,2,\ldots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}^+$ the set of positive real numbers. Finally, the notation $h_v \sim g_v$ means that $\lim_{v\to 0} h_v/g_v = 1$.

II. Problem Formulation

Consider a dynamic system with a Markov evolution. $X_i, i \in \mathbb{N}$, is the state vector at time i and $\mathcal{S}$ is the state space, with cardinality M. In particular, $\{X_i\}_{i\in\mathbb{N}}$ forms a discrete-time, homogeneous Markov chain with given initial distribution π and transition probabilities

$$a(x_i, x_j) = P\big(\{X_k = x_i\}|\{X_{k-1} = x_j\}\big), \quad x_i, x_j \in \mathcal{S}.$$

A sequence of states $\{X_i\}_{i=1}^{k}$, often called trajectory, is denoted with $\boldsymbol{X}_{1:k}$ and has density $p_k(\boldsymbol{x}_{1:k}) = \pi(x_1)\prod_{i=2}^{k} a(x_{i-1}, x_i)$, with respect to the counting measure. $\{X_i\}_{i\in\mathbb{N}}$ is observed through a set of noisy measurements. The measurement process is $\{Z_i\}_{i\in\mathbb{N}}$, and the sample space of each Z_i is $(\mathcal{Z}, \mathfrak{Y})$, $\mathfrak{Y}$ being a σ-algebra of subsets of $\mathcal{Z}$. Consider a σ-finite measure ν on $(\mathcal{Z}, \mathfrak{Y})$. If the signal $\{X_i\}_{i\in\mathbb{N}}$ is present, $\{(X_i, Z_i)\}_{i\in\mathbb{N}}$ is an HMM: given a realization $\{x_i\}_{i\in\mathbb{N}}$ of $\{X_i\}_{i\in\mathbb{N}}$, $\{Z_i\}_{i\in\mathbb{N}}$ is a sequence of conditionally independent random variables, each Z_i having density $f(z|x_i)$ with respect to ν. On the other hand, if the measurements contain only noise, $\{Z_i\}_{i\in\mathbb{N}}$ is an i.i.d. process, each Z_i having density $f(z|\theta_0)$ with respect to ν. Thus, for every $k \in \mathbb{N}$, the joint distribution of $\boldsymbol{Z}_{1:k}$ has conditional density

$$f_k(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k}) = \prod_{i=1}^{k} f(z_i|x_i), \quad \text{if the signal is present and } \boldsymbol{X}_{1:k} = \boldsymbol{x}_{1:k}$$

$$f_k(\boldsymbol{z}_{1:k}|\theta_0) = \prod_{i=1}^{k} f(z_i|\theta_0), \qquad \text{if the signal is not present}$$

with respect to ν^k.

Given these elements, one is to sample the process $\{Z_i\}_{i\in\mathbb{N}}$ sequentially and decide, as soon as possible, if measurements are generated by noise alone or if they come from a dynamic system. In the latter case, it can be also required to estimate the system trajectory which has generated such measurements. The parameter space, then, is $\{\theta_0\} \cup (\times_{i\in\mathbb{N}}\mathcal{S})$. As in [1]–[3], whose focus, however, was on nonsequential decision rules, there is a mutual coupling of detection and estimation and two different strategies may be adopted. Indeed, the structure of the decision rule can be chosen so as to improve the detection or the estimation performance. The former case is called a weakly coupled (or uncoupled) design while the latter a strongly coupled (or coupled) design. In both cases, the estimator is enabled by the detection operation: this gating, however, can be (possibly) optimal for the detection or for the estimation. However, the problem of designing sequential procedures for detection and estimation is considerably more difficult than that of devising FSS procedures [30] and the approach taken in general is to extend and generalize the SPRT designing a practical, possibly suboptimal, rule [11], [12], [14], [15], [21]–[23]. In this paper, the uncoupled strategy is adopted, this choice being motivated by a number of reasons: it has a very simple structure: it exhibits many optimal properties, as shown in Section IV; detection is the primary interest in many practical applications, as for example, radar surveillance problems which will be discussed later.

III. Detection and Estimation Procedure

A sequential decision rule is the pair (φ, ξ), where $\varphi = \{\varphi_k\}_{k\in\mathbb{N}}$ is a stopping rule and $\xi = \{\xi_k\}_{k\in\mathbb{N}}$ a terminal decision rule [29]. Since detection and estimation are performed in parallel, the terminal decision rule is itself composed of a detection rule $\delta = \{\delta_k\}_{k\in\mathbb{N}}$ for testing the signal presence and of a trajectory estimator $\widehat{x} = \{\widehat{x}_k\}_{k\in\mathbb{N}}$, i.e., $\xi = (\delta, \widehat{x})$. The proposed (nonrandomized) sequential decision rule is then

$$\varphi_k(\boldsymbol{z}_{1:k}) = \begin{cases} 1, & \text{if } \Lambda_k(\boldsymbol{z}_{1:k}) \notin (\gamma_0, \gamma_1) \\ 0, & \text{otherwise} \end{cases} \tag{1a}$$

$$\delta_k(\boldsymbol{z}_{1:k}) = \begin{cases} 1, & \text{if } \Lambda_k(\boldsymbol{z}_{1:k}) \ge \gamma_1 \\ 0, & \text{if } \Lambda_k(\boldsymbol{z}_{1:k}) \le \gamma_0 \end{cases} \tag{1b}$$

$$\widehat{x}_k(\boldsymbol{z}_{1:k}) = \arg\max_{\boldsymbol{x}_{1:k}\in\mathcal{S}^k} p_k(\boldsymbol{x}_{1:k})\Lambda_k(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k}), \quad \text{if } \Lambda_k(\boldsymbol{z}_{1:k}) \ge \gamma_1 \tag{1c}$$

where $\Lambda_k(\boldsymbol{z}_{1:k}) = \sum_{\boldsymbol{x}_{1:k}\in\mathcal{S}^k} p_k(\boldsymbol{x}_{1:k})\Lambda_k(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k})$, $\Lambda_k(\cdot|\boldsymbol{x}_{1:k})$ being the likelihood ratio of $f_k(\cdot|\boldsymbol{x}_{1:k})$ to $f_k(\cdot|\theta_0)$.

Notice that the pair (φ,δ) is an SPRT for testing $H_0 =$ "noise only" against the alternative $H_1 =$ "signal present," no matter of its trajectory. H_1, then, is the hypothesis that $\boldsymbol{Z}_{1:k}$ has density

$$f_{k,H_1}(\boldsymbol{z}_{1:k}) = \sum_{\boldsymbol{x}_{1:k}\in\mathcal{S}^k} p_k(\boldsymbol{x}_{1:k}) f_k(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k}), \quad \forall k\in\mathbb{N}.$$

The strength of such a sequential test is the pair of probabilities of errors of the first and second kind, α and β, respectively (often, in detection problems, α is referred to as probability of false alarm, $P_{\rm fa}$, and β as probability of miss, $P_{\rm miss}$). Denoting with τ the stopping time and with $\psi = \{\psi_k\}_{k\in\mathbb{N}}$ its conditional distribution,[1] these probabilities of error are given by

$$\alpha = \sum_{k\in\mathbb{N}} \mathrm{E}_{\theta_0}[\psi_k(\boldsymbol{Z}_{1:k})\delta_k(\boldsymbol{Z}_{1:k})]$$
$$\beta = \sum_{k\in\mathbb{N}} \mathrm{E}_{H_1}[\psi_k(\boldsymbol{Z}_{1:k})(1-\delta_k(\boldsymbol{Z}_{1:k}))],$$

$P_{\rm d} = \sum_{k\in\mathbb{N}} \mathrm{E}_{H_1}[\psi_k(\boldsymbol{Z}_{1:k})\delta_k(\boldsymbol{Z}_{1:k})]$ being the probability of detection.[2] The boundaries of the test, γ_0 and γ_1, with $0<\gamma_0<1<\gamma_1<+\infty$, are chosen in order to have the required strength (α,β).

Concerning $\hat{x}$ only, it can be considered a gated estimator since estimation is enabled by the detection rule. Furthermore, consider the triplet $(\tau, \boldsymbol{X}_{1:\tau}, \boldsymbol{Z}_{1:\tau})$. Since

$$\begin{aligned}
&P(\{\tau=k, \boldsymbol{X}_{1:\tau}=\boldsymbol{x}_{1:k}, \boldsymbol{Z}_{1:\tau}\in A_k\}|\{\tau<+\infty\\
&\delta_\tau(\boldsymbol{Z}_{1:\tau})=1, H_1\})\\
&\quad=\frac{P(\{\tau=k, \boldsymbol{X}_{1:\tau}=\boldsymbol{x}_{1:k}, \boldsymbol{Z}_{1:\tau}\in A_k, \delta_\tau(\boldsymbol{Z}_{1:\tau})=1\}|H_1)}{P(\{\tau<+\infty, \delta_\tau(\boldsymbol{Z}_{1:\tau})=1\}|H_1)}\\
&\quad=P_{\rm d}^{-1}\int_{A_k} p(\boldsymbol{x}_{1:k}) f_k(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k})\psi_k(\boldsymbol{z}_{1:k})\delta_k(\boldsymbol{z}_{1:k})d\nu^k(\boldsymbol{z}_{1:k}),\\
&\qquad\qquad \forall k\in\mathbb{N},\ \boldsymbol{x}_{1:k}\in\mathcal{S}^k,\ \text{and } A_k\in\sigma(\boldsymbol{Z}_{1:k}),
\end{aligned}$$

$P_{\rm d}^{-1}p(\boldsymbol{x}_{1:k})f(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k})\psi_k(\boldsymbol{z}_{1:k})\delta_k(\boldsymbol{z}_{1:k})$ is the density of $(\tau,\boldsymbol{X}_{1:\tau},\boldsymbol{Z}_{1:\tau})|\{$ accept H_1, H_1 true$\}$. This means that $P_{\rm d}^{-1}\sum_{k\in\mathbb{N}} p(\boldsymbol{x}_{1:k})f(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k})\psi_k(\boldsymbol{z}_{1:k})\delta_k(\boldsymbol{z}_{1:k})$ is the density of $(\boldsymbol{X}_{1:\tau},\boldsymbol{Z}_{1:\tau})|$ $\{$accept H_1, H_1 true$\}$ so that

$$\arg\max_{\theta\in\times_{i\in\mathbb{N}}\mathcal{S}} \sum_{k\in\mathbb{N}} p(\boldsymbol{x}_{1:k})f(\boldsymbol{Z}_{1:k}|\boldsymbol{x}_{1:k})\psi_k(\boldsymbol{Z}_{1:k})\delta_k(\boldsymbol{Z}_{1:k})$$

is a maximum *a posteriori* probability (MAP) estimator conditioned upon the event $\{$accept H_1, H_1 true$\}$, $\boldsymbol{x}_{1:k}$ being the projection of θ on $\mathcal{S}^k$. Since the above estimation rule is exactly that in (1c), it results that $\hat{x}$ is a MAP estimation rule conditioned upon the event that no error of the first kind is made by the detector.

Finally, as to the computational complexity, the sequential decision rule in (1) requires to evaluate the statistics

$$\sum_{\boldsymbol{x}_{1:k}\in\mathcal{S}^k} p_k(\boldsymbol{x}_{1:k})\Lambda_k(\boldsymbol{Z}_{1:k}|\boldsymbol{x}_{1:k})$$

$$\max_{\boldsymbol{x}_{1:k}\in\mathcal{S}^k} p_k(\boldsymbol{x}_{1:k})\Lambda_k(\boldsymbol{Z}_{1:k}|\boldsymbol{x}_{1:k}), \quad \text{for } k=1,\ldots,\tau$$

where

$$p_k(\boldsymbol{x}_{1:k})\Lambda_k(\boldsymbol{Z}_{1:k}) = \pi(x_1)\frac{f(Z_1|x_1)}{f(Z_1|\theta_0)}\prod_{i=2}^k a(x_{i-1},x_i)\frac{f(Z_i|x_i)}{f(Z_i|\theta_0)}.$$

These statistics, then, have the form of a stage-separated function on the algebraic system $(\mathbb{R},+,\cdot)$ and $(\mathbb{R},\max,\cdot)$, respectively, and can be computed through two dynamic programming algorithms [31] (similar to the forward–backward procedure and to the Viterbi algorithm as in [32]), whereby the computational complexity is only linear in k. Notice that maximization in (1c) is preferred to $\arg\max_{\boldsymbol{x}_{1:k}\in\mathcal{S}^k} p_k(\boldsymbol{x}_{1:k})f_k(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k})$ since, in the former case, the estimator can work on the same data as the detector, thus further lowering the complexity.

IV. Asymptotic Analysis

Let $\mathfrak{T}(\alpha',\beta')$, $\alpha',\beta'\in(0,1)$, be the class of nonrandomized tests (denoted with (N,d), N being the stopping time and d the terminal decision rule), either sequential or FSS, with probability of error of the first and second kind bounded by α' and β', respectively. It is known that Wald's SPRT for testing a simple hypothesis against a simple alternative based on i.i.d. observations has the following remarkable properties [33], [34], [28], [29].

(i) If the test has strength (α,β) and boundaries γ_0,γ_1, then

$$\alpha\le(1-\beta)/\gamma_1\le 1/\gamma_1 \quad\text{and}\quad \beta\le(1-\alpha)\gamma_0\le\gamma_0. \tag{2}$$

(ii) The test ends a.s. under both hypotheses.

(iii) The ASN is finite under both hypotheses.

(iv) The ASN is minimized among tests in the class $\mathfrak{T}(\alpha',\beta')$ under both hypotheses.

Except property (i), which is easily shown to hold under very general conditions [35], the other properties in general do not hold in the present setting since the observations $\{Z_i\}_{i\in\mathbb{N}}$ are not independent. This section is devoted to studying the asymptotic behavior of the sequential test when the two error probabilities simultaneously approach zero. It will be demonstrated that, under rather mild hypotheses, the procedure satisfies also properties (ii) and (iii) and, asymptotically, (iv). In particular, it will be shown that every finite moment of the stopping time is first-order asymptotically minimized in the class $\mathfrak{T}(\alpha',\beta')$. The regularity conditions are stated below.

Condition 4.1: The Markov chain $\{X_k\}_{k\in\mathbb{N}}$ is stationary, irreducible and aperiodic.

Condition 4.2: The family of mixtures of at most M elements of $\{f(\cdot|x)\}_{x\in\mathcal{S}}$ is not equal to $f(\cdot|\theta_0)$ ν-a.e., i.e., for every distribution c on $\mathcal{S}$, $\sum_{x\in\mathcal{S}} c(x)f(\cdot|x)$ and $f(\cdot|\theta_0)$ are not equal ν-a.e.

Condition 4.3: For every $x\in\mathcal{S}$, $\mathrm{E}_{H_i}[|\ln\frac{f(Z_1|x)}{f(Z_1|\theta_0)}|]<+\infty$, $i=0,1$.

Condition 4.4: There exists a constant $a>0$ such that

$$\mathrm{E}_{H_0}\left[\left(\frac{f(Z_1|x)}{f(Z_1|\theta_0)}\right)^{\pm a}\right] \quad\text{and}\quad \mathrm{E}_y\left[\left(\frac{f(Z_1|x)}{f(Z_1|\theta_0)}\right)^{\pm a}\right]$$

are finite, for all $x,y\in\mathcal{S}$.

[1] $\psi_k(\boldsymbol{z}_{1:k})$ is the probability that $\tau=k$ given a realization $\boldsymbol{z}_{1:k}$ of $\boldsymbol{Z}_{1:k}$, for any $k\in\mathbb{N}$; the relationship between ψ and φ is $\psi_1(z_1)=\varphi_1(z_1)$ and $\psi_k(\boldsymbol{z}_{1:k})=\varphi(\boldsymbol{z}_{1:k})\prod_{\ell=1}^{k-1}(1-\varphi(\boldsymbol{z}_{1:\ell}))$, for $k>1$.

[2] Notice that $\beta+P_{\rm d}=P(\{\tau<+\infty\}|H_1)$.

Condition 4.5: $(\frac{f(z|x)}{f(z|\theta_0)})^{\pm 1} \neq 0$ for every $x \in \mathcal{S}$ and $z \in \mathcal{Z}$.

Condition 4.6: The matrix containing the transition probabilities $\{a(x,y)\}_{x,y\in\mathcal{S}}$ is invertible.

Remark 4.7: Since the Markov chain $\{X_i\}_{i\in\mathbb{N}}$ is homogeneous and has a finite state space, Condition 4.1 corresponds to requiring that $\{X_i\}_{i\in\mathbb{N}}$ be stationary and ergodic, which will be seen to imply $\{Z_i\}_{i\in\mathbb{N}}$ to be stationary and ergodic as well, an essential property for the limiting theorem to be presented. It can be shown (recursively) that Condition 4.2 ensures that the two densities $f_k(\cdot|\theta_0)$ and $f_{k,H_1}(\cdot)$ are not ν-a.e. equal for every $k \in \mathbb{N}$: otherwise, for some $k \in \mathbb{N}$ it could not be possible to discriminate between statistical populations drawn from these two distributions, i.e., detection could not be possible. Finally, Conditions 4.3–4.6 are essentially "regularity" conditions which allow to derive the limiting behavior of the log-likelihood ratios $\{\ln \Lambda_k\}_{k\in\mathbb{N}}$. Notice, furthermore, that Condition 4.4 implies Condition 4.3 since $|\ln w| \leq C(w^a + w^{-a})$ for any $w > 0$ and sufficiently large C.

It turns out that the validity of properties (ii)–(iv) is highly influenced by the limits $\lambda_i = \lim_{k\to+\infty} k^{-1}\mathrm{E}_{H_i}[\ln \Lambda_k(\boldsymbol{Z}_{1:k})]$, $i = 0, 1$, in the case that they exist finite and nonzero [20]. For this reason, we first give the following (see the Appendix for proof).

Theorem 4.8: If Conditions 4.1–4.3 are fulfilled, there exist finite constants $\lambda_0 < 0$ and $\lambda_1 > 0$ such that

$$\lim_{k\to+\infty} \frac{1}{k}\mathrm{E}_{H_i}[\ln \Lambda_k(\boldsymbol{Z}_{1:k})] = \lambda_i$$
$$\lim_{k\to+\infty} \frac{1}{k} \ln \Lambda_k(\boldsymbol{Z}_{1:k}) = \lambda_i, \text{a.s. under } H_i, \qquad i = 0, 1.$$

These conclusions hold for any initial probability with strictly positive entries (i.e., not necessarily the stationary one) used in the definition of $\{\Lambda_k\}_{k\in\mathbb{N}}$ and λ_i have the same value.

If the log-likelihood ratios $\{\ln \Lambda_k(\boldsymbol{Z}_{1:k})\}_{k\in\mathbb{N}}$ satisfy this sort of "stability" property, we can easily demonstrate the following.

Proposition 4.9: Under Conditions 4.1–4.3, the test ends a.s. under both hypotheses, i.e., $P(\{\tau < +\infty\}|H_i) = 1, i = 1, 0$.

Proof: Define the two auxiliary stopping times

$$\tau_0 = \inf\{k \in \mathbb{N} : \Lambda_k(\boldsymbol{Z}_{1:k}) \leq \gamma_0\}$$
$$\tau_1 = \inf\{k \in \mathbb{N} : \Lambda_k(\boldsymbol{Z}_{1:k}) \geq \gamma_1\}.$$

From Theorem 4.8, $\lim_{k\to+\infty} k^{-1} \ln \Lambda_k(\boldsymbol{Z}_{1:k}) = \lambda_i$, a.s. under H_i, $i = 0, 1$, with $\lambda_0 < 0$ and $\lambda_1 > 0$. This implies that

$$P(\{\lim_{k\to+\infty} \Lambda_k(\boldsymbol{Z}_{1:k}) < \gamma_0\}|H_0) = 1$$
$$P(\{\lim_{k\to+\infty} \Lambda_k(\boldsymbol{Z}_{1:k}) > \gamma_1\}|H_1) = 1$$

which means that $P(\{\tau_i < +\infty\}|H_i) = 1, i = 0, 1$. The thesis, then, follows form the fact that $\tau = \min\{\tau_0, \tau_1\}$. □

The "stability" of $\{\ln \Lambda_k(\boldsymbol{Z}_{1:k})\}_{k\in\mathbb{N}}$ of Theorem 4.8 is sufficient also to imply the asymptotic optimality of the test in the sense of the following statement.

Theorem 4.10: Suppose that Conditions 4.1–4.3 are fulfilled, γ_0 and γ_1 are chosen so that the test belongs to $\mathfrak{T}(\alpha', \beta')$, and $\ln \gamma_1 \sim \ln \frac{1}{\alpha'}$, $\ln \gamma_0 \sim \ln \beta'$ as $\alpha' + \beta' \to 0$. Then

$$\frac{\tau}{|\ln \beta'|} \underset{[\alpha'+\beta'\to 0]}{\longrightarrow} \frac{1}{|\lambda_0|}, \quad \text{a.s. under } H_0$$
$$\frac{\tau}{|\ln \alpha'|} \underset{[\alpha'+\beta'\to 0]}{\longrightarrow} \frac{1}{\lambda_1}, \quad \text{a.s. under } H_1.$$

Furthermore, for every $\varepsilon \in (0, 1)$, we get

$$\inf_{(N,d)\in\mathfrak{T}(\alpha',\beta')} P(\{N > \varepsilon\tau\}|H_i) \underset{\alpha'+\beta'\to 0}{\longrightarrow} 1, \quad i = 0, 1. \quad (3)$$

Proof: Under Conditions 4.1–4.3, the conclusions of Theorem 4.8 hold and then Theorem 1 of [20] can be used. □

Notice that (2) allows to choose the appropriate thresholds: indeed, it implies that $\gamma_1 = 1/\alpha'$ and $\gamma_0 = \beta'$ result in a test which belongs to $\mathfrak{T}(\alpha', \beta')$. Moreover, regarding (2) as approximate equalities (i.e., by neglecting overshoots) leads to the approximations (see [33])

$$\gamma_1 \approx (1-\beta)/\alpha \overset{\alpha+\beta\to 0}{\sim} 1/\alpha, \quad \gamma_0 \approx \beta/(1-\alpha) \overset{\alpha+\beta\to 0}{\sim} \beta. \quad (4)$$

Notice that (3) does not imply asymptotic optimality of the test (the optimality criterion of the Wald–Wolfwitz theorem [34] is about the minimization of the ASN under both hypotheses). With the a.s. convergence of $\{k^{-1} \ln \Lambda_k(\boldsymbol{Z}_{1:k})\}_{k\in\mathbb{N}}$ alone, the following can be proved.

Theorem 4.11: If conditions 4.1–4.3 are satisfied then

$$\liminf_{\alpha'+\beta'\to 0} \inf_{(N,d)\in\mathfrak{T}(\alpha',\beta')} \frac{\mathrm{E}_{H_0}[N^r]}{|\ln \beta'|^r} \geq \frac{1}{|\lambda_0|^r}$$
$$\liminf_{\alpha'+\beta'\to 0} \inf_{(N,d)\in\mathfrak{T}(\alpha',\beta')} \frac{\mathrm{E}_{H_1}[N^r]}{|\ln \alpha'|^r} \geq \frac{1}{\lambda_1^r}$$

for every positive constant r.

Proof: It can be shown similarly to [21, Theorem 2.2] applying the Markov's inequality and Theorem 4.10. □

In order to guarantee finiteness of the expected sample size and to obtain its first-order asymptotic minimization, Condition 4.3 must be strengthened requiring Conditions 4.4–4.6 to hold. Indeed, the following can be proved (proof is given in the Appendix).

Theorem 4.12: Suppose that Conditions 4.1, 4.2, 4.4–4.6 are fulfilled, γ_0 and γ_1 are chosen so that the test belongs to $\mathfrak{T}(\alpha', \beta')$, and $\ln \gamma_1 \sim \ln \frac{1}{\alpha'}$, $\ln \gamma_0 \sim \ln \beta'$ as $\alpha' + \beta' \to 0$. Then, for every $r \in \mathbb{N}$, $\mathrm{E}_{H_i}[\tau^r] < +\infty$, $i = 0, 1$, and, as $\alpha' + \beta' \to 0$

$$\inf_{(N,d)\in\mathfrak{T}(\alpha',\beta')} \mathrm{E}_{H_0}[N^r] \sim \mathrm{E}_{H_0}[\tau^r] \sim \frac{|\ln \beta'|^r}{|\lambda_0|^r}$$
$$\inf_{(N,d)\in\mathfrak{T}(\alpha',\beta')} \mathrm{E}_{H_1}[N^r] \sim \mathrm{E}_{H_1}[\tau^r] \sim \frac{|\ln \alpha'|^r}{\lambda_1^r}.$$

The $\mathcal{O}(1)$ term is due to the overshoot $\ln(\Lambda_\tau(\boldsymbol{Z}_{1:\tau})/\gamma_0)$ or $\ln(\Lambda_\tau(\boldsymbol{Z}_{1:\tau})/\gamma_1)$, analogous to Wald's lower bound [33] for the sample size (which is attained when these overshoots are ignored).

From Theorems 4.10 and 4.12, the asymptotic behavior of the test is determined by the constants λ_i, $i=0,1$. These constants are often difficult to evaluate and, thus, approximations or bounds can be useful. To this end, the following propositions, whose proofs are given in the Appendix, are presented.

Proposition 4.13: Constants λ_i, $i=0,1$, satisfy the following:

$$\lambda_1 \le \sum_{x\in\mathcal{S}} \overline{\pi}(x) D(f(\cdot|x)\|f(\cdot|\theta_0)) \tag{5a}$$

$$|\lambda_0| \le \sum_{x\in\mathcal{S}} \overline{\pi}(x) D(f(\cdot|\theta_0)\|f(\cdot|x)) \tag{5b}$$

where $\overline{\pi}$ is the unique stationary distribution of the Markov chain $\{X_i\}_{i\in\mathbb{N}}$.

Proposition 4.14: If for every distribution c on $\mathcal{S}$

$$\mathrm{D}\left(\sum_{x\in\mathcal{S}} c(x) f(\cdot|x)\|f(\cdot|\theta_0)\right) = \mathrm{D}\left(\sum_{x\in\mathcal{S}} c'(x) f(\cdot|x)\|f(\cdot|\theta_0)\right) \tag{6a}$$

$$\mathrm{D}\left(f(\cdot|\theta_0)\|\sum_{x\in\mathcal{S}} c(x) f(\cdot|x)\right) = \mathrm{D}\left(f(\cdot|\theta_0)\|\sum_{x\in\mathcal{S}} c'(x) f(\cdot|x)\right) \tag{6b}$$

for any permutation c' of c, then

$$D\left(\frac{1}{M}\sum_{y\in\mathcal{S}} f(\cdot|y)\|f(\cdot|\theta_0)\right) \le \lambda_1 \le D(f(\cdot|x)\|f(\cdot|\theta_0)) \tag{7a}$$

$$D\left(f(\cdot|\theta_0)\|\frac{1}{M}\sum_{y\in\mathcal{S}} f_1(\cdot|y)\right) \le |\lambda_0| \le D(f(\cdot|\theta_0)\|f(\cdot|x)) \tag{7b}$$

where $D(f(\cdot|x)\|f(\cdot|\theta_0))$ and $D(f(\cdot|\theta_0)\|f(\cdot|x))$ assume the same value for any $x\in\mathcal{S}$.

Notice that the upper bounds in (7b) are attained if $\pi(x)=1$ for some $x\in\mathcal{S}$ and $a(x,x)=1, \forall x\in\mathcal{S}$, while the lower bounds if $\pi(x)=1/M, \forall x\in\mathcal{S}$, and $a(x,y)=1/M, \forall x,y\in\mathcal{S}$.

V. Example of Application: The Radar Case

The radar problem is characterized by the inherent presence of multiple-resolution elements, which correspond to range "bins" as well as Doppler, azimuth, and elevation cells. This problem has been solved in [11], [16], [24] but all of these approaches concern the case that the target is not allowed to change its position while being illuminated by the radar. This condition may be too restrictive, especially in airborne applications where the relative radial velocity between target and radar may exceed Mach 2. The surveillance area is divided into smaller angular regions, each visited in turn by the antenna beam in cyclic manner. In each region, a sequential procedure is used to accept or reject the hypothesis that a single target is present. The measurement process is obtained dividing the region into a grid and discretizing the continuous-time received signal accordingly (if the grid is sufficiently fine, losses due to possible mismatches may be neglected). The measurement at epoch $i\in\mathbb{N}$, also called frame, is the set of returns received from all of the radar resolution elements, i.e., $Z_\ell = \{Z_\ell(x) : x \in \{1,\dots,N_a\}\times\{1,\dots,N_e\}\times\{1,\dots,N_r\}\times\{1,\dots,N_d\}\}$, where N_a, N_e, N_r, N_d are the number of resolution elements in azimuth, elevation, range, and Doppler, respectively. The target signature appears on at most one resolution element in each frame. The target state space consists of the set of all the resolution cells, i.e., $\mathcal{S}=\{1,\dots,N_a\}\times\{1,\dots,N_e\}\times\{1,\dots,N_r\}\times\{1,\dots,N_d\}$, with $M = N_aN_eN_rN_d$. If also velocities are to be considered, then the state space can be enlarged consequently. A first-order Gaussian–Markov random-walk model is used to derive the transition probabilities, which are given by $a(x_i,x_{i+1}) = \prod_{\ell=1}^4 a_\ell(x_{i,\ell},x_{i+1,\ell})$, where $x_{i,\ell}$ denotes the ℓth component of the target state vector at epoch i and

$$a_\ell(x_{i,\ell},x_{i+1,\ell}) = Q\left(\frac{x_{i+1,\ell}-x_{i,\ell}-1/2}{\sigma_\ell}\right) - Q\left(\frac{x_{i+1,\ell}-x_{i,\ell}+1/2}{\sigma_\ell}\right),\ \ell=1,\dots,4.$$

In the preceding equation, $Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-t^2/2}dt$ and σ_ℓ is a parameter related to the target mobility along the ℓth dimension: large values of σ_ℓ allow large target maneuvers but decrease, at the same time, target detection and estimation capabilities.[3] In reference to the initial probability, if no other prior information is available (for example, previous detections), it is reasonable to force $\pi(x)=1/M$, for all $x\in\mathcal{S}$.

It is supposed that the components of the measurement Z_i are independent, each $Z_i(x)$, $x\in\mathcal{S}$, being an exponentially distributed random variable with density

$$h_1(v) = \frac{e^{-\frac{v}{1+\rho}}}{1+\rho}\mathrm{u}(v), \text{if the target is present in location } x \tag{8a}$$

$$h_0(v) = e^{-v}\mathrm{u}(v), \text{otherwise} \tag{8b}$$

where ρ denotes the signal-to-noise ratio (SNR) and $\mathrm{u}(y)$ is the unit-step function. In this case

$$f(z_i|x_i) = h_1(z_i(x_i)) \prod_{\substack{x\in\mathcal{S}\\ x\ne x_i}} h_0(z_i(x)),$$

$$f(z_i|\theta_0) = \prod_{x\in\mathcal{S}} h_0(z_i(x)),$$

and[4]

$$\Lambda_k(\boldsymbol{z}_{1:k}) = \prod_{i=1}^k \frac{h_1(z_i(x_i))}{h_0(z_i(x_i))} = \prod_{i=1}^k \frac{e^{z_i(x_i)\rho/(1+\rho)}}{1+\rho} \tag{9}$$

$\forall k\in\mathbb{N}$. Notice that, as can be easily checked, this model satisfies Conditions 4.2–4.6 and, also (6). In particular, for Condition 4.4 to hold, it is sufficient to choose $a < 1/\rho$. This means

[3]Notice that, even if all transitions are theoretically admissible, real targets necessarily need to satisfy physical constraints, such as limitations on the maximum velocity and acceleration. In this case, a truncated Gaussian density can be used.

[4]The model of (8)–(9) applies, for examples, if measurements come from a square law envelope detector; the noise is additive, white, and Gaussian; the target has a Swerling-I fluctuation model, and frequency agility is used to achieve frame-to-frame target amplitude decorrelation. This is a common situation in radar scenarios [36].

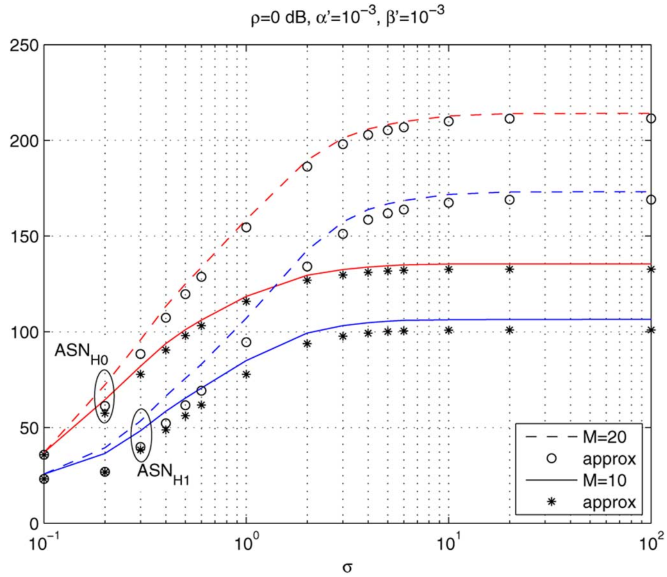

Fig. 1. ASN under both hypotheses versus the mobility parameter σ for $\rho = 0$ dB and different values of M. The markers denote the values resulting from the asymptotic approximations of the ASN of Theorem 4.12.

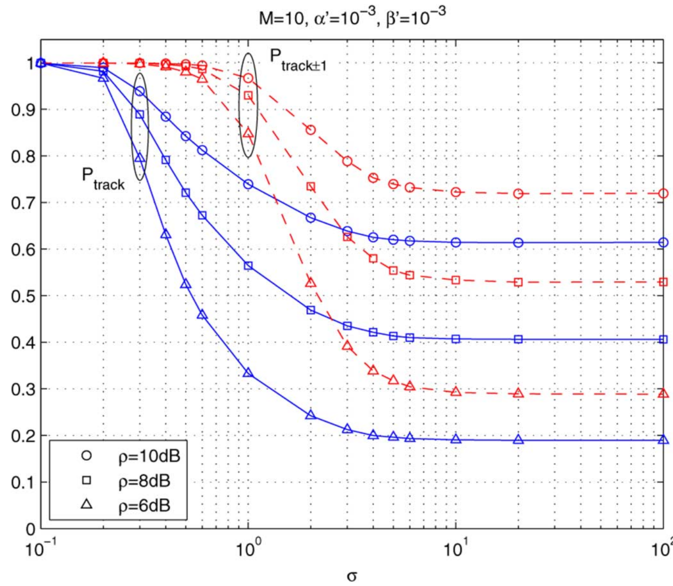

Fig. 2. Probabilities of correct track estimation P_{track} and $P_{\text{track}\pm1}$ versus the mobility parameter σ for different values of ρ and for $M = 11$.

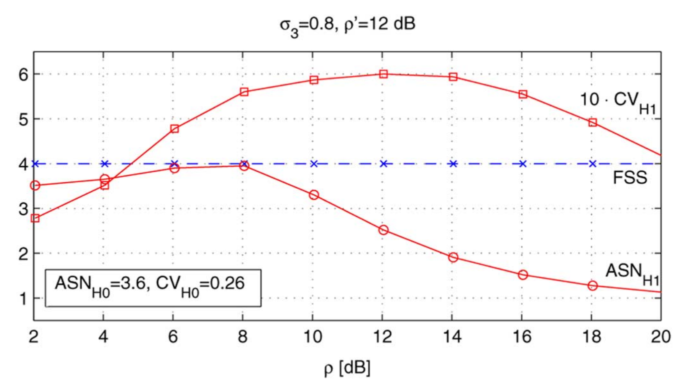

Fig. 3. ASN and coefficient of variation versus the SNR under H_1. The dotted line represent the sample size of the equivalent FSS rule.

that the test first order asymptotically minimizes any positive moment of the stopping time distribution. Even in the above situation, however, occasionally long observations can be needed. Furthermore, if there are mismatches between design and actual values of some parameters (for example, the SNR) the resulting ASN can be very large, especially for small error probabilities. Truncation of the procedure then can be used to prevent such a problem: when a fixed sample K is reached, hypothesis H_1 or H_0 is accepted whether $\Lambda_K(\boldsymbol{Z}_{1:K})$ exceeds γ_K or not, respectively. The impact of truncation on the system performances as well as the problem of the final threshold setting is not explored further here and the reader is referred to earlier literature [37]–[39].

VI. Numerical Results

The behavior of the sequential procedure has been tested through Monte Carlo simulations in terms of ASN and P_{track}, the probability of correct track estimation. First, a general problem of detection and trajectory estimation is considered in order to reinforce the discussion in Sections III and IV. For simplicity, the measurement model is that of (8)–(9), even if there is no explicit reference to a radar scenario. The state space is $\mathcal{S} = \{1, \ldots, M\}$ and the transition probabilities are derived from a truncated Gaussian distribution with standard deviation σ using a quantization step of 10^{-4}. The boundaries γ_0 and γ_1 have been set using (4), where the design error probabilities $\alpha' = \beta' = 10^{-3}$ have been adopted. In Fig. 1, the ASN under both H_0 and H_1 is plotted versus σ for $\rho = 0$ dB and for various M. Since the model satisfies Conditions 4.1–4.6, the approximations for the ASN of Theorem 4.12 hold: the difference between the approximations and the true ASN is due to the excesses of $\ln \Lambda_\tau(\boldsymbol{Z}_{1:\tau})$ over boundaries. Furthermore, (6) are fulfilled so that the bounds on λ_0 and λ_1 of Proposition 4.14 hold and the extrema of the ASN are (asymptotically) reached for $a(x,x) = 1 \ \forall x \in \mathcal{S}$ and $a(x,y) = 1/M \ \forall x,y \in \mathcal{S}$ ($\sigma = 10^{-1}$ and $\sigma = 10^4$ given the adopted quantization step). It is confirmed then the intuitive idea that more compact priors allows easier detections. Fig. 2 shows the effect of the SNR on the probability of correct track estimation. $P_{\text{track}\pm1}$, the probability that the distance between each state of the recovered trajectory and the actual state is less then or equal to 1, has also been plotted. Notice that, since an uncoupled design has been adopted, the estimation performances decrease as ρ is lowered and/or σ is increased while α and β are not influenced by these parameters (indeed the lower values of ρ and/or larger values of σ are traded for larger ASNs).

The remaining curves concern more specifically the radar scenario outlined in Section V. The search zone is composed of a single elevation and four azimuth sectors; the other parameters are $N_r = 100$ and $N_d = 16$. The transition probabilities along the third dimension (i.e., range) are defined as above, with a maximum admissible range transition of ± 3 bins. Azimuth transitions are neglected while Doppler ones are assumed equally likely (i.e., $\sigma_2 = 0$ and $\sigma_4 = +\infty$). The truncation stage is $K = 20$, with $\gamma_K = \sqrt{\gamma_0 \gamma_1}$, while the nominal SNR

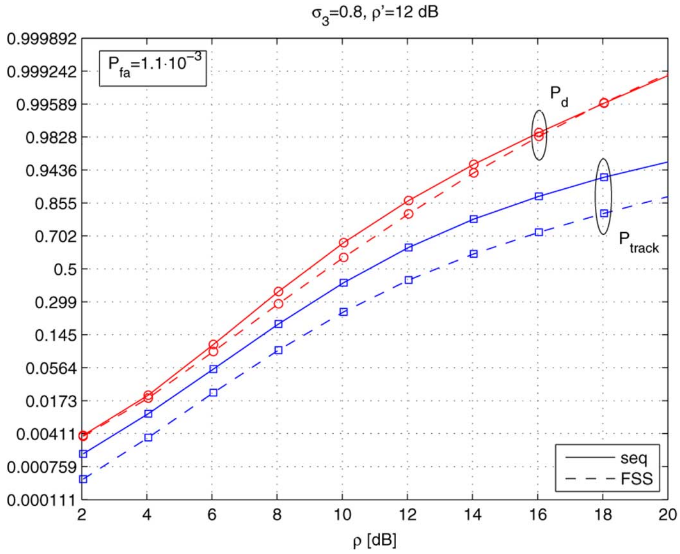

Fig. 4. Probability of detection and correct trajectory estimation versus the SNR for both the sequential and the FSS procedures. Vertical axis in Gaussian scale.

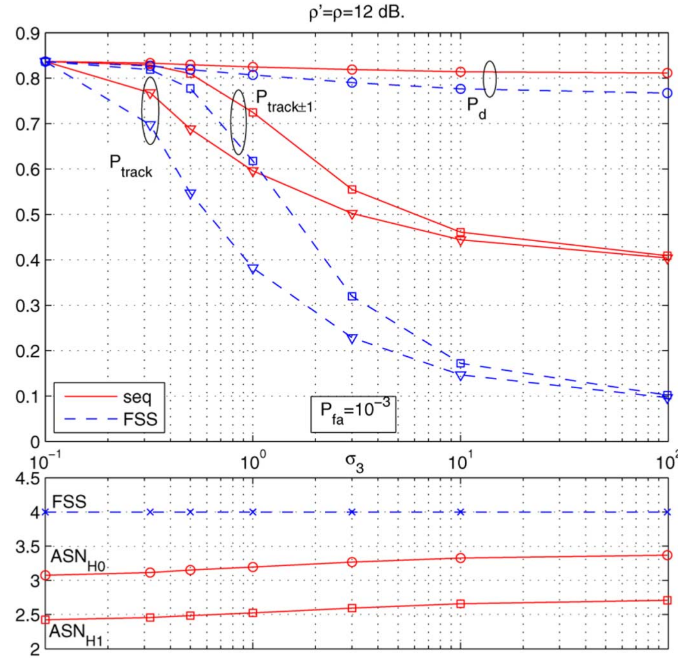

Fig. 5. Probabilities of detection and trajectory estimation versus the range target mobility parameter for both sequential and FSS procedures (top); ASN in the lower plot.

has been set at $\rho' = N_d$[5] (notice that, since it is not realistic to assume prior knowledge of the target strength, the actual SNR is in general different from the design value ρ'). The subsequent plots are aimed both at assessing the effect of a mismatch between ρ and ρ' and at giving a comparison with an equivalent FSS procedure exhibiting the same $P_{\rm fa}$. In Fig. 3, the ASN and the coefficient of variation (CV) of the sample size[6] are represented versus ρ under H_1. Notice the characteristic peak at intermediate values of the SNR: yet the effect of the beam antenna remaining blocked monitoring a particular direction has been avoided by truncation. As for the FSS rule, a conservative choice has been made taking four samples, which is uniformly larger than the ASN of the truncated sequential procedure. Fig. 4 shows $P_{\rm d}$ and $P_{\rm track}$ versus the SNR for both the sequential and the FSS procedure. It can be seen that the sequential procedure achieve larger $P_{\rm d}$ over all the inspected range of SNRs maintaining, at the same time, a full sample size saving. Notice also the massive gain granted in terms of $P_{\rm track}$ which is mainly due to the low ASN required. Finally, Fig. 5 shows the performances in terms of the target mobility. $P_{\rm d}$, $P_{\rm track}$, $P_{\rm track\pm1}$, and the ASN are represented versus σ_3 for $\rho = \rho'$ (recall that $\sigma = 10^{-1}$ corresponds to the case of a steady target). It can be seen, that, while the probability of detection of the FSS procedure impairs as the target mobility increases, that of the sequential rule remains almost unchanged, in that large values of σ_3 are counterbalanced by higher ASNs. As to the estimation performance, it obviously decreases in both cases, but sequential techniques retain their superiority over all the range of σ_3.

VII. Conclusion

The general problem of sequential detection and possible trajectory estimation of a dynamic system observed through a set of noisy measurements has been considered. Previous limitation on the system dynamics imposed by other works present in the literature have been removed and a thorough analysis of the asymptotic behavior of the test has been presented. In particular, it has been shown that, under rather mild conditions, the test is asymptotically optimal, in the sense that it minimizes, up to a $\mathcal{O}(1)$ term, any moment of the stopping time distribution under both hypothesis as the probabilities of error approach zero. Possible applications to radar surveillance problems have been inspected. Finally, the numerical results have confirmed the correctness of the given approximations and have demonstrated the merits of the proposed strategy with respect to other competitors in the context of radar surveillance.

Appendix

In the following, the proofs of Theorems 4.8, 4.12 and Propositions 4.13, 4.14 are given.

Proof of Theorem 4.8: Part of the proof borrows its arguments from [40]. Since Condition 4.1 is satisfied, $\{X_i\}_{i\in\mathbb{N}}$ is stationary and ergodic and, from [40, Lemma 1], $\{Z_i\}_{i\in\mathbb{N}}$ is stationary and ergodic as well. Given the one-sided stationary process $\{(X_i, Z_i)\}_{i\in\mathbb{N}}$, it is extended to a two-sided stationary process $\{(X_i, Z_i)\}_{i\in\mathbb{Z}}$ in the usual way.

Define

$$q_k(\boldsymbol{z}_{1:k}|x) = \frac{f(z_1|x)}{f(z_1|\theta_0)} \sum_{x_2\in\mathcal{S}} \cdots \sum_{x_k\in\mathcal{S}} a(x, x_2) \frac{f(z_2|x_2)}{f(z_2|\theta_0)} \cdot \prod_{i=3}^{k} a(x_{i-1}, x_i) \frac{f(z_i|x_i)}{f(z_i|\theta_0)}$$

$$q_k(\boldsymbol{z}_{1:k}) = \max_{x\in\mathcal{S}} q_k(\boldsymbol{z}_{1:k}|x)$$

[5]If each frame results from processing pulse trains of N_d pulses, $\rho' = N_d$ corresponds to an SNR per pulse of 0 dB.

[6]The CV of a random variable is the ratio σ/μ of its standard deviation σ and its mean $\mu \neq 0$.

with $q_1(z_1|x) = \frac{f(z_1|x)}{f(z_1|\theta_0)}$. Since $q_k(\boldsymbol{z}_{1:k}|x)$ is the likelihood ratio $\Lambda_k(\boldsymbol{z}_{1:k})$ given that $X_1 = x$, it results, $\forall \boldsymbol{z}_{1:k} \in \mathcal{Z}^k$ and $k \in \mathbb{N}$, in

$$\Lambda_k(\boldsymbol{z}_{1:k}) \leq q_k(\boldsymbol{z}_{1:k})$$
$$\Lambda_k(\boldsymbol{z}_{1:k}) = \sum_{x\in\mathcal{S}} \pi(x) q_k(\boldsymbol{z}_{1:k}|x) \geq \max_{x\in\mathcal{S}}\{\pi(x) q_k(\boldsymbol{z}_{1:k}|x)\}$$
$$\geq \max_{x\in\mathcal{S}}\{\min_{y\in\mathcal{S}}\{\pi(y)\} q_k(\boldsymbol{z}_{1:k}|x)\} = \min_{x\in\mathcal{S}}\{\pi(x)\} q_k(\boldsymbol{z}_{1:k}).$$

Combining the above inequalities results in

$$\frac{1}{k}\ln \min_{x_1\in\mathcal{S}}\{\pi(x_1)\} \leq \frac{1}{k}\ln\frac{\Lambda_k(\boldsymbol{z}_{1:k})}{q_k(\boldsymbol{z}_{1:k})} \leq 0,$$

$\forall \boldsymbol{z}_{1:k} \in \mathcal{Z}^k$ and $k \in \mathbb{N}$, which implies that

$$\lim_{k\to+\infty}\frac{1}{k}\ln\Lambda_k(\boldsymbol{Z}_{1:k}) = \lim_{k\to+\infty}\frac{1}{k}\ln q_k(\boldsymbol{Z}_{1:k}), \text{ a.s. under } H_i$$
$$\lim_{k\to+\infty}\frac{1}{k}\mathrm{E}_{H_i}[\ln\Lambda_k(\boldsymbol{Z}_{1:k})] = \lim_{k\to+\infty}\frac{1}{k}\mathrm{E}_{H_i}[\ln q_k(\boldsymbol{Z}_{1:k})]$$

for $i = 0, 1$. As a consequence, it is sufficient to demonstrate the conclusions of the theorem for $q_k(\boldsymbol{Z}_{1:k})$. The advantage of working with q_k rather than Λ_k is twofold. First, q_k does not depend upon the initial probability π and, then, the second part of the theorem is demonstrated. The second advantage results from the following relationship:

$$q_{s+t}(\boldsymbol{z}_{1:s+t}) \leq q_s(\boldsymbol{z}_{1:s}) q_t(\boldsymbol{z}_{s+1:s+t}), \quad \forall s,t \geq 1 \tag{A1}$$

for any sequence $\{z_i\}_{i\in\mathbb{N}}$ (the proof is identical to that of [10, Lemma 3]). Define now a doubly indexed sequence of random variables $\{W_{st}\}_{t>s\geq 0}$ by $W_{st} = \ln q_{t-s}(\boldsymbol{Z}_{s+1:t})$. With this definition, the stochastic process $\{W_{st}\}_{t>s\geq 0}$ satisfies the following three properties.

1) From (A1), $W_{st} \leq W_{su} + W_{ut}$, $\forall s < u < t$, i.e., it is a subadditive process.
2) By the stationarity of $\{Z_k\}_{k\in\mathbb{Z}}$, $\{W_{st}\}_{t>s\geq 0}$ is stationary relative to the shift transformation $W_{st} \to W_{s+1t+1}$, i.e., W_{st}, and W_{s+1t+1} have the same distribution.
3) By Condition 4.3, for $i = 0, 1$, we have

$$\mathrm{E}_{H_i}[W_{01}^+] = \mathrm{E}_{H_i}\left[\max_{x\in\mathcal{S}}\left(\ln\frac{f(Z_1|x)}{f(Z_1|\theta_0)}\right)^+\right] \leq \sum_{x\in\mathcal{S}}\mathrm{E}_{H_i}\left[\left|\ln\frac{f(Z_1|x)}{f(Z_1|\theta_0)}\right|\right] < +\infty$$

where $v^+ = \max\{0, v\}$, $v \in \mathbb{R}$.

By Kingman's subadditive ergodic theorem [4, Theorems 1.5 and 1.8], a stochastic process satisfying these properties also satisfies the conclusions of the ergodic theorem, i.e.,

i) $\lim_{k\to+\infty} k^{-1} W_{0k} = W < +\infty$ exists almost surely;
ii) $\mathrm{E}[W] = \lim_{k\to+\infty} k^{-1}\mathrm{E}[W_{0k}]$;
iii) W is degenerate if the process is ergodic.

Thus, an application to $W_{0k} = \ln q_k(\boldsymbol{Z}_{1:k})$ gives (ergodicity carries over from $\{Z_k\}_{k\in\mathbb{N}}$) that

$$\lim_{k\to+\infty}\frac{1}{k}\mathrm{E}_{H_i}[\ln q_k(\boldsymbol{Z}_{1:k})] = \lambda_i < +\infty, \quad i = 0, 1$$

exists and

$$\lim_{k\to+\infty}\frac{1}{k}\ln q_k(\boldsymbol{Z}_{1:k}) = \lambda_i, \quad \text{a.s. under } H_i, i = 0, 1.$$

Since $k^{-1}\ln\Lambda_k(\boldsymbol{Z}_{1:k})$ and $k^{-1}\ln q_k(\boldsymbol{Z}_{1:k})$ have the same limiting behavior, the proof of the theorem is complete if one demonstrates that λ_0 is finite, strictly negative, and that λ_1 is strictly positive.

In order to prove that λ_0 is bounded also from below, first notice that $\Lambda_k(\boldsymbol{z}_{1:k})$ is a convex combination of the terms $f_k(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k})/f_k(\boldsymbol{z}_{1:k}|\theta_0)$, for any $\boldsymbol{z}_{1:k} \in \mathcal{Z}^k$ and $k \in \mathbb{N}$, and thus

$$\ln\Lambda_k(\boldsymbol{z}_{1:k}) \geq \ln\min_{\boldsymbol{x}_{1:k}\in\mathcal{S}^k}\frac{f_k(\boldsymbol{z}_{1:k}|\boldsymbol{x}_{1:k})}{f_k\boldsymbol{z}_{1:k}|\theta_0)} = \sum_{n=1}^{k}\min_{x\in\mathcal{S}}\ln\frac{f(z_n|x)}{f(z_n|\theta_0)} = \sum_{n=1}^{k}\eta(z_n)$$

where $\{\eta(Z_k)\}_{k\in\mathbb{N}}$ forms a sequence of i.i.d. random variables under H_0. Furthermore, by Condition 4.3,

$$\mathrm{E}_{H_0}[|\eta(Z_1)|] = \mathrm{E}_{H_0}\left[\left|\min_{x\in\mathcal{S}}\ln\frac{f(Z_1|x)}{f(Z_1|\theta_0)}\right|\right] < +\infty,$$

which implies that

$$\lambda_0 = \lim_{k\to+\infty}\frac{1}{k}\mathrm{E}_{H_0}[\ln\Lambda_k(\boldsymbol{Z}_{1:k})] \geq \mathrm{E}_{H_0}[\eta(Z_1)] > -\infty.$$

As for the sign of λ_i, $i = 0, 1$, let $g_n(z_n|\boldsymbol{z}_{1:n-1})$ denote the conditional density given by the ratio $f_{n,H_1}(\boldsymbol{z}_{1:n})/f_{n-1,H_1}(\boldsymbol{z}_{1:n-1})$ for $n \geq 2$ and by $f_{1,H_1}(z_1)$ for $n = 1$. With this notation, the limiting constants λ_i are also given by

$$\lambda_i = \lim_{k\to+\infty}\frac{1}{k}\mathrm{E}_{H_i}[\ln\Lambda_k(\boldsymbol{Z}_{1:k})] = \lim_{k\to+\infty}\frac{1}{k}\sum_{n=1}^{k}\mathrm{E}_{H_i}\left[\ln\frac{g_n(Z_n|\boldsymbol{Z}_{1:n-1})}{f(Z_n|\theta_0)}\right] = \lim_{k\to+\infty}\frac{1}{k}\sum_{n=1}^{k}\mathrm{E}_{H_i}\left[\ln\frac{g_n(Z_1|\boldsymbol{Z}_{-n+2:0})}{f(Z_1|\theta_0)}\right], \quad i = 0, 1 \tag{A2}$$

where stationarity of $\{Z_i\}_{i\in\mathbb{Z}}$ has been exploited. On the other hand, $g_n(Z_1|\boldsymbol{Z}_{-n+2:0})$ can be written as

$$\sum_{x\in\mathcal{S}} f(Z_1|x) P(\{X_1 = x\}|\boldsymbol{Z}_{-n+2:0}), \quad \forall n \in \mathbb{N}$$

and this implies that

$$\min_{x\in\mathcal{S}}\frac{f(Z_1|x)}{f(Z_1|\theta_0)} \leq \frac{g_n(Z_1|\boldsymbol{Z}_{-n+2:0})}{f(Z_1|\theta_0)} \leq \max_{x\in\mathcal{S}}\frac{f(Z_1|x)}{f(Z_1|\theta_0)},$$

$n \in \mathbb{N}$ and, by Condition 4.3, that $\{\ln\frac{g_n(Z_1|\boldsymbol{Z}_{-n+2:0})}{f(Z_1|\theta_0)}\}_{n\in\mathbb{N}}$ is a sequence of uniformly integrable random variables. In this case, dominated convergence gives

$$\lim_{n\to+\infty}\mathrm{E}_{H_i}\left[\ln\frac{g_n(Z_1|\boldsymbol{Z}_{-n+2:0})}{f(Z_1|\theta_0)}\right] = \mathrm{E}_{H_i}\left[\ln\frac{g(Z_1|\boldsymbol{Z}_{-\infty:0})}{f(Z_1|\theta_0)}\right] \tag{A3}$$

$i = 0, 1$. In the above equation, $g(Z_1|\boldsymbol{Z}_{-\infty:0})$ denotes the limit

$$\lim_{n\to+\infty} g_n(Z_1|\boldsymbol{Z}_{-n+2:0}) = \lim_{n\to+\infty}\sum_{x\in\mathcal{S}} f(Z_1|x) P(\{X_1 = x\}|\boldsymbol{Z}_{-n+2:0}) = \sum_{x\in\mathcal{S}} f(Z_1|x) P(\{X_1 = x\}|\boldsymbol{Z}_{-\infty:0}) \tag{A4}$$

where the latter equality follows form a martingale convergence theorem by Lévy (see [42]). Finally, from (A3) and from the Cesáro mean theorem it follows that λ_i in (A2) can be also written as

$$\lambda_i = \mathrm{E}_{H_i}\left[\ln \frac{g(Z_1|\boldsymbol{Z}_{-\infty:0})}{f(Z_1|\theta_0)}\right], \quad i=0,1$$

which implies that

$$\lambda_1 = \mathrm{E}_{H_1}\left[\mathrm{D}(g(\cdot|\boldsymbol{Z}_{-\infty:0})\|f(\cdot|\theta_0))\right] > 0 \tag{A5a}$$
$$\lambda_0 = -\mathrm{E}_{H_0}[\mathrm{D}(f(\cdot|\theta_0)\|g(\cdot|\boldsymbol{Z}_{-\infty:0}))] < 0. \tag{A5b}$$

Inequalities in (A5) result from the fact that the Kullback–Leibler divergence is always nonnegative and is equal to zero if and only if the two densities are equal ν-a.e.: this, however, cannot happen since, from (A4), $g(\cdot|\{\boldsymbol{Z}_{-\infty:0} = \boldsymbol{z}_{-\infty:0}\})$ is always a mixture of M elements of $\{f(\cdot|x)\}_{x\in\mathcal{S}}$, which is not ν-a.e. equal to $f(\cdot|\theta_0)$ by Condition 4.2. □

Proof of Theorem 4.12: Exploiting the idea introduced in [26], [43], the likelihood ratio can be equivalently represented as $\Lambda_k(\boldsymbol{Z}_{1:k}) = \|\underline{\underline{M}}_k\underline{\pi}\|$, $k\in\mathbb{N}$, where $\|\cdot\|$ is the L_1-norm on $\mathbb{R}^M$ and $\underline{\underline{M}}_k$ is an $M\times M$ matrix on $\mathbb{R}$ defined as follows:

$$\underline{\underline{M}}_1 = \underline{\underline{T}}_1, \quad \underline{\underline{M}}_k = \underline{\underline{T}}_k \underline{\underline{A}}^T \underline{\underline{M}}_{k-1}, \quad \text{for } k \geq 2$$

with $(\cdot)^T$ denoting transpose, $\underline{\underline{T}}_k$ being a diagonal $M\times M$ matrix with entries $\{f(Z_k|x)/f(Z_k|\theta_0)\}_{x\in\mathcal{S}}$, $\underline{\underline{A}} = [a(x,y)]_{x,y\in\mathcal{S}}$ the transition probability matrix, and $\underline{\pi} = [\pi(x)]_{x\in\mathcal{S}}$ the initial probability vector. Under Conditions 4.5 and 4.6, $\underline{\underline{M}}_k$ is invertible for every $\boldsymbol{z}_{1:k}\in\mathcal{Z}^k$, $k\in\mathbb{N}$, while, under hypothesis H_1, $\{X_k, Z_k\}_{k\in\mathbb{N}}$ is a Markov chain on $\mathcal{S}\times\mathcal{Z}$. This implies that the process $\{(X_k,Z_k),\underline{\underline{M}}_k\}_{k\in\mathbb{N}}$ is a multiplicative Markovian process (see [44, Definition 1.1]) and $\{\underline{\underline{M}}_k\}_{k\in\mathbb{N}}$ a product of Markov random matrices.

In order to exploit the large deviations result for products of Markov random matrices in [44], we first need to verify the validity of conditions A of [26] and [44]. Under Condition 4.1, $\{X_k\}_{k\in\mathbb{N}}$ is uniformly ergodic and so is the Markov chain $\{(X_k, Z_k)\}_{k\in\mathbb{N}}$ (see [45]), i.e., condition A_1 is fulfilled. As concerns A_2, it is requested that there exists $p>0$ such that

$$\mathrm{E}\left[e^{p\sup\left\{\ln\left\|\underline{\underline{M}}_k\right\|,\ln\left\|\underline{\underline{M}}_k^{-1}\right\|\right\}} | X_1 = x_1, Z_1 = z_1\right] < +\infty \tag{A6}$$

$\forall (x_1,z_1)\in\mathcal{S}\times\mathcal{Z}$ and $k=0,1$, where the expectation is taken with respect to the joint distribution of $\{X_k,Z_k\}_{k\in\mathbb{N}}$ and the matrix norm is that induced by the vector norm, i.e., $\|\underline{\underline{M}}_k\| = \sup_{\|\vec{u}\|=1}\|\underline{\underline{M}}_k\vec{u}\|$. Since

$$e^{p\sup\{\ln\|\underline{\underline{M}}_2\|,\ln\|\underline{\underline{M}}_2^{-1}\|\}} \leq \|\underline{\underline{T}}_2\underline{\underline{A}}^T\underline{\underline{T}}_1\|^p + \|(\underline{\underline{T}}_2\underline{\underline{A}}^T\underline{\underline{T}}_1)^{-1}\|^p$$
$$\leq \left(\|\underline{\underline{A}}^T\|\|\underline{\underline{T}}_1\|\|\underline{\underline{T}}_2\|\right)^p + \left(\|(\underline{\underline{A}}^T)^{-1}\|\|\underline{\underline{T}}_1^{-1}\|\|\underline{\underline{T}}_2^{-1}\|\right)^p$$

with

$$\|\underline{\underline{T}}_k^i\|^p = \left(\max_{x\in\mathcal{S}}\left(\frac{f(Z_k|x)}{f(Z_k|\theta_0)}\right)^i\right)^p \leq \sum_{x\in\mathcal{S}}\left(\frac{f(Z_k|x)}{f(Z_k|\theta_0)}\right)^{ip}, \quad i=\pm 1.$$

(A6) is satisfied if

$$\mathrm{E}\left[\left(\frac{f(Z_1|w)}{f(Z_1|\theta_0)}\frac{f(Z_2|x)}{f(Z_2|\theta_0)}\right)^{\pm p} | X_1 = x_1, Z_1 = z_1\right] < +\infty, \quad \forall w,x,x_1\in\mathcal{S} \text{ and } z_1\in\mathcal{Z}.$$

But

$$\begin{aligned}
&\mathrm{E}\left[\left(\frac{f(Z_1|w)}{f(Z_1|\theta_0)}\frac{f(Z_2|x)}{f(Z_2|\theta_0)}\right)^{\pm p} | X_1 = x_1, Z_1 = z_1\right]\\
&= \left(\frac{f(z_1|w)}{f(z_1|\theta_0)}\right)^{\pm p}\sum_{y\in\mathcal{S}}\int_{\mathcal{Z}}\left(\frac{f(z|x)}{f(z|\theta_0)}\right)^{\pm p}\\
&\quad\times a(x_1,y)f(z|y)\nu(dz)\\
&= \left(\frac{f(z_1|w)}{f(z_1|\theta_0)}\right)^{\pm p}\sum_{y\in\mathcal{S}}a(x_1,y)\mathrm{E}_y\left[\left(\frac{f(Z_1|x)}{f(Z_1|\theta_0)}\right)^{\pm p}\right]
\end{aligned}$$

which is bounded for every $(w,z_1)\in\mathcal{S}\times\mathcal{Z}$ by Condition 4.5 and for every $x_1,x,y\in\mathcal{S}$ by Condition 4.4. In reference to Condition A_3, the process $\{(X_k,Z_k),\underline{\underline{M}}_k\}_{k\in\mathbb{N}}$ has to be strongly irreducible and contracting (see [26, Definition 2] for the terminology) and this can be shown using arguments similar to those in [26, proof of Proposition 4].

Since these three conditions are satisfied, the result is that

(i) there exists a neighborhood I of the origin such that, for every $p\in I$, $(\mathrm{E}[\|\underline{\underline{M}}_k\underline{\pi}\|^p | X_1 = x_1, Z_1 = z_1])^{1/k}$ converges to a function $H(p)$ uniformly in $(x_1,z_1)\in\mathcal{S}\times\mathcal{Z}$ [44, Sec. 4 and Theorem 4.3];

(ii) $H'(0) = \lambda_1$ [44, Proposition 3.8].

These two properties will be used to derive the convergence rate of the sequence $\{k^{-1}\ln\Lambda_k(\boldsymbol{Z}_{1:k})\}_{k\in\mathbb{N}}$. To this end, define the function

$$\begin{aligned}
\widetilde{H}(p) &= \limsup_{k\to+\infty}\frac{1}{k}\ln \mathrm{E}_{H_1}\left[e^{p\ln\Lambda_k(\boldsymbol{Z}_{1:k})}\right]\\
&= \limsup_{k\to+\infty}\frac{1}{k}\ln\mathrm{E}_{H_1}\left[\Lambda_k^p(\boldsymbol{Z}_{1:k})\right].
\end{aligned}$$

Given uniform convergence in (x_1,z_1) and recalling that $\|\underline{\underline{M}}_k\underline{\pi}\| = \Lambda_k(\boldsymbol{Z}_k)$, property (i) implies that

$$\lim_{k\to+\infty}\frac{1}{k}\ln\mathrm{E}_{H_1}\left[\Lambda_k^p(\boldsymbol{Z}_{1:k})\right] = \ln H(p), \quad \forall p\in I$$

and then $\widetilde{H}(p) = \ln H(p)$, $\forall p\in I$, so that, from property (ii), $\widetilde{H}'(0) = \lambda_1$. Denote now with $\mathcal{D}_{\widetilde{H}}$ the set $\{p\in\mathbb{R} : \widetilde{H}(p) < +\infty\}$. Since, from Condition 4.4

$$\begin{aligned}
\mathrm{E}_{H_1}\left[\Lambda_k^a(\boldsymbol{Z}_{1:k})\right] &\leq \mathrm{E}_{H_1}\left[\prod_{n=1}^k \max_{x\in\mathcal{S}}\left(\frac{f(Z_n|x)}{f(Z_n|\theta_0)}\right)^a\right]\\
&\leq \left(\sum_{y\in\mathcal{S}}\sum_{x\in\mathcal{S}}\mathrm{E}_y\left[\left(\frac{f(Z_1|x)}{f(Z_1|\theta_0)}\right)^a\right]\right)^k < +\infty
\end{aligned}$$

it follows that $\widetilde{H}(\pm a) < +\infty$ and, thus, the interior part of $\mathcal{D}_{\widetilde{H}}$ contains the point $p=0$. This and the fact that $\widetilde{H}'(0) = \lambda_1$ implies that $\{\frac{1}{k}\ln\Lambda_k(\boldsymbol{Z}_{1:k})\}_{k\in\mathbb{N}}$ converges to λ_1 exponentially in

k [46, Exercise 2.3.25], [47, Theorem IV.1].[7] The exponential convergence is obviously much stronger than the a.s. convergence granted by Theorem 4.8. Indeed, the former implies that

$$\sum_{k=1}^{+\infty} k^r P\left(\{|k^{-1}\ln\Lambda_k(\boldsymbol{Z}_{1:k}) - \lambda_1| \geq \varepsilon\}|H_1\right) < +\infty, \quad \forall\,\varepsilon > 0 \text{ and } r > 0$$

which in turn implies that $\{k^{-1}\ln\Lambda_k(\boldsymbol{Z}_{1:k})\}_{k\in\mathbb{N}}$ converges r-quickly to λ_1 for any $r > 0$ [48, Lemma 3].[8] The latter is also called strong complete convergence in [21], [22]. Given the r-quick convergence for any $r > 0$, since λ_1 is finite and strictly positive from Theorem 4.8, the statement follows from [20, Corollary 1].

The case under H_0 can be handled similarly considering $\{Z_k, \underline{\underline{M}}_k\}_{k\in\mathbb{N}}$, where $\{Z_k\}_{k\in\mathbb{N}}$ is now an i.i.d. process. □

In order to prove Propositions 4.13 and 4.14, the following lemma is first needed.

Lemma A.1: $\mathrm{D}(\sum_{x\in\mathcal{S}} c(x) f(\cdot|x)\|f(\cdot|\theta_0))$ and $\mathrm{D}(f(\cdot|\theta_0)\|\sum_{x\in\mathcal{S}} c(x)f(\cdot|x))$ are convex function on the set

$$\left\{\{c(x)\}_{x\in\mathcal{S}} \in [0,1]^M : \sum_{x\in\mathcal{S}} c(x) = 1\right\}.$$

Proof: It can be verified exploiting Jensen's inequality and convexity of functions $-\ln v$ and $v\ln v$, $v \in \mathbb{R}^+$. □

Proof of Proposition 4.13: From the proof of Theorem 4.8, (A4), and (A5), λ_1 can be also written as

$$\mathrm{E}_{H_1}\left[\mathrm{D}\left(\sum_{x\in\mathcal{S}} f(\cdot|x)P(\{X_1 = x\}|\boldsymbol{Z}_{-\infty:0})\|f(\cdot|\theta_0)\right)\right].$$

On the other hand, Lemma A.1 and Jensen's inequality allow us to write

$$\begin{aligned}\mathrm{D}&\left(\sum_{x\in\mathcal{S}} f(\cdot|x)P(\{X_1 = x\}|\{\boldsymbol{Z}_{-\infty:0} = \boldsymbol{z}_{-\infty:0}\})\|f(\cdot|\theta_0)\right)\\ &\leq \sum_{x\in\mathcal{S}} P(\{X_1 = x\}|\{\boldsymbol{Z}_{-\infty:0} = \boldsymbol{z}_{-\infty:0}\})\mathrm{D}(f(\cdot|x)\|f(\cdot|\theta_0))\end{aligned}$$

for every realization $\boldsymbol{z}_{-\infty:0}$ of $\boldsymbol{Z}_{-\infty:0}$, whereby

$$\begin{aligned}\lambda_1 &\leq \mathrm{E}_{H_1}\left[\sum_{x\in\mathcal{S}} P(\{X_1 = x\}|\boldsymbol{Z}_{-\infty:0})\mathrm{D}(f(\cdot|x)\|f(\cdot|\theta_0))\right]\\ &= \sum_{x\in\mathcal{S}} \mathrm{E}_{H_1}[P(\{X_1 = x\}|\boldsymbol{Z}_{-\infty:0})]\mathrm{D}(f(\cdot|x)\|f(\cdot|\theta_0))\\ &= \sum_{x\in\mathcal{S}} \overline{\pi}(x)D(f(\cdot|x)\|f(\cdot|\theta_0)).\end{aligned}$$

The upper bound on $|\lambda_0|$ can be proved similarly. □

Proof of Proposition 4.14: From (6), setting $c(x) = 1$ for some $x \in \mathcal{S}$, it follows that $\mathrm{D}(f(\cdot|x)\|f(\cdot|\theta_0))$ has the same value for every $x \in \mathcal{S}$: this, along with Proposition 4.13, demonstrates the upper bound on λ_1. As for the lower bound, exploiting Lemma A.1 and Jensen's inequality, it follows that

$$\begin{aligned}\mathrm{D}&\left(\sum_{x\in\mathcal{S}} c(x)f(\cdot|x)\|f(\cdot|\theta_0)\right)\\ &= \frac{1}{M!}\sum_{i=1}^{M!}\mathrm{D}\left(\sum_{x\in\mathcal{S}} c_i'(x)f(\cdot|x)\|f(\cdot|\theta_0)\right)\\ &\geq \mathrm{D}\left(\sum_{x\in\mathcal{S}} f(\cdot|x)\frac{1}{M!}\sum_{i=1}^{M!} c_i'(x)\|f(\cdot|\theta_0)\right)\\ &= \mathrm{D}\left(\sum_{x\in\mathcal{S}} \frac{1}{M} f(\cdot|x)\|f(\cdot|\theta_0)\right)\end{aligned} \tag{A7}$$

for every probability vector c on $\mathcal{S}$, where $\{c_i'\}_{i=2}^{M!}$ is the set of all of the possible permutations of c and $c_1' = c$. From the demonstration of Theorem 4.8, (A4), and (A5), λ_1 can be written also as

$$\mathrm{E}_{H_1}\left[\mathrm{D}\left(\sum_{x\in\mathcal{S}} f(\cdot|x)P(\{X_1 = x\}|\boldsymbol{Z}_{-\infty:0})\|f(\cdot|\theta_0)\right)\right]$$

and thus, exploiting (A7), it results in

$$\begin{aligned}\lambda_1 &\geq \mathrm{E}_{H_1}\left[\mathrm{D}\left(\sum_{x\in\mathcal{S}} \frac{1}{M} f(\cdot|x)\|f(\cdot|\theta_0)\right)\right]\\ &= \mathrm{D}\left(\sum_{x\in\mathcal{S}} \frac{1}{M} f(\cdot|x)\|f(\cdot|\theta_0)\right)\end{aligned}$$

and the lower bound is proved. The bounds on $|\lambda_0|$ can be proved similarly. □

[7] A sequence $\{Y_k\}_{k\in\mathbb{N}}$ of random variables is said to converge exponentially to a constant λ if, for any sufficiently small $\epsilon > 0$, there exists a constant C such that $P(|Y_k - \lambda| \geq \epsilon) \leq e^{-kC}$ [47].

[8] A sequence $\{Y_k\}_{k\in\mathbb{N}}$ of random variables is said to converge r-quickly to a constant λ, for some $r > 0$, if $\mathrm{E}[T_\epsilon^r] < +\infty$, for all $\epsilon > 0$, where $T_\epsilon = \sup\{k \in \mathbb{N} : |Y_k - \lambda| \geq \epsilon\}(\sup\{\emptyset\} = 0)$ [49].

References

[1] D. Middleton and R. Esposito, "Simultaneous optimum detection and estimation of signals in noise," *IEEE Trans. Inf. Theory*, vol. 14, no. 3, pp. 434–444, May 1968.

[2] A. Fredriksen, D. Middleton, and D. Vandelinde, "Simultaneous signal detection and estimation under multiple hypotheses," *IEEE Trans. Inf. Theory*, vol. IT-18, no. 5, pp. 607–614, Sep. 1972.

[3] B. Baygün and A. O. Hero, "Optimal simultaneous detection and estimation under a false alarm constraint," *IEEE Trans. Inf. Theory*, vol. 41, no. 3, pp. 688–703, May 1995.

[4] A. S. Willsky, "A survey of design methods for failure detection in dynamic systems," *Automatica*, vol. 12, no. 6, pp. 601–611, Nov. 1976.

[5] M. A. Massoumnia, G. C. Verghese, and A. S. Willsky, "Failure detection and identification," *IEEE Trans. Automat. Contr.*, vol. 34, no. 3, pp. 316–321, Mar. 1989.

[6] H. Im and T. Kim, "Optimization of multiframe target detection schemes," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 35, no. 1, pp. 176–187, Jan. 1999.

[7] N. Merhav and Y. Ephraim, "A bayesian classification approach with application to speech recognition," *IEEE Trans. Signal Process.*, vol. 39, no. 10, pp. 2157–2166, Oct. 1991.

[8] Y. Barniv, "Dynamic programming solution for detecting dim moving targets," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 21, no. 1, pp. 144–156, Jan. 1985.

[9] J. Arnold, S. Shaw, and H. Pasternack, "Efficient target tracking using dyamic programming," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 29, no. 1, pp. 44–56, Jan. 1993.

[10] S. M. Tonissen and R. J. Evans, "Performance of dynamic progrmming techniques for track-before-detect," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 32, no. 4, pp. 1440–1450, Oct. 1996.

[11] M. B. Marcus and P. Swerling, "Sequential detection in radars with multiple resolution elements," *IRE Trans. Inf. Theory*, vol. IT-8, no. 3, pp. 237–245, Apr. 1962.

[12] P. Armitage, "Sequential analysis with more than two alternative hypotheses and its relation to discriminant function analysis," *J. Roy. Statist. Soc. Ser. B*, vol. 12, no. 1, pp. 250–263, 1950.

[13] J. Kiefer and J. Sacks, "Asymptotically optimum sequential inference and design," *Ann. Math. Statist.*, vol. 34, no. 3, pp. 705–750, Sep. 1963.

[14] G. Lorden, "Likelihood ratio tests for sequential k-decision problems," *Ann. Math. Statist.*, vol. 43, no. 5, pp. 1412–1427, Oct. 1972.

[15] G. Lorden, "Nearly-optimal sequential test for finitely many parameter values," *Ann. Statist.*, vol. 5, no. 1, pp. 1–21, Jan. 1977.

[16] C. W. Helstrom, "A range-sampled sequential detection system," *IRE Trans. Inf. Theory*, vol. IT-8, no. 1, pp. 43–47, Jan. 1962.

[17] I. S. Reed and I. Selin, "A sequential test for the presence of a signal in one of k possible positions," *IEEE Trans. Inf. Theory*, vol. IT-9, no. 4, pp. 286–288, Oct. 1963.

[18] C. W. Baum and V. Veeravalli, "A sequential procedure for multihypothesis testing," *IEEE Trans. Inf. Theory*, vol. 40, no. 6, pp. 1994–2007, Nov. 1994.

[19] S. Marano, V. Matta, and P. Willett, "Sequential detection of almost-harmonic signals," *IEEE Trans. Signal Process.*, vol. 51, no. 2, pp. 395–406, Feb. 2003.

[20] T. L. Lai, "Asymptotic optimality of invariant sequential probability ratio tests," *Ann. Statist.*, vol. 9, no. 2, pp. 318–333, Mar. 1981.

[21] A. G. Tartakovsky, "Asymptotic optimality of certain multihypothesis sequential tests: Non-i.i.d. case," *Statist. Inference for Stochastic Processes*, vol. 1, no. 3, pp. 265–295, Oct. 1998.

[22] V. Dragalin, A. G. Tartakovsky, and V. Veeravalli, "Multihypothesis sequential probability ratio test—Part I: Asymptotic optimality," *IEEE Trans. Inform. Theory*, vol. 45, no. 7, pp. 2448–2461, Nov. 1999.

[23] V. Dragalin, A. G. Tartakovsky, and V. Veeravalli, "Multihypothesis sequential probability ratio test—Part II: Accurate asymptotic expansions for the expected sample size," *IEEE Trans. Inf. Theory*, vol. 46, no. 4, pp. 1366–1383, July 2000.

[24] A. G. Tartakovsky, X. R. Li, and G. Yaralov, "Sequential detection of targets in multichannel systems," *IEEE Trans. Inf. Theory*, vol. 49, no. 2, pp. 425–445, Feb. 2003.

[25] Y. Ephraim and N. Merhav, "Hidden Markov processes," *IEEE Trans. Inf. Theory*, vol. 48, no. 6, pp. 1518–1569, Jun. 2002.

[26] C. D. Fuh, "Sprt and cusum in hidden markov models," *Ann. Statist.*, vol. 31, pp. 942–977, Apr. 2003.

[27] B. Chen and P. Willett, "Detection of hidden markov model transient signals," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 36, no. 4, pp. 1253–1268, Oct. 2000.

[28] D. L. Burkholder and R. A. Wijsman, "Optimum properties and admissibility of sequential tests," *Ann. Math. Statist.*, vol. 34, no. 1, pp. 1–17, Mar. 1963.

[29] T. S. Ferguson, *Mathematical Statistics—A Decision-Theoretic Approach*. New York: Academic, 1967.

[30] D. Blackwell and M. A. Girshick, *Theory of Games and Statistical Decisions*. New York: Wiley, 1954.

[31] S. Verdú and H. V. Poor, "Abstract dynamic programming models under commutativity conditions," *SIAM J. Contr. Optimiz.*, vol. 25, no. 4, pp. 990–1006, Jul. 1987.

[32] E. Grossi and M. Lops, "Sequential along-track integration for early detection of moving targets," *IEEE Trans. Signal Process.*, vol. 56, no. 8, pt. 2, pp. 3969–3982, Aug. 2008.

[33] A. Wald, "Sequential tests of statistical hypotheses," *Ann. Math. Statist.*, vol. 16, no. 2, pp. 117–186, June 1945.

[34] A. Wald and J. Wolfowitz, "Optimum character of the sequential probability ratio test," *Ann. Math. Statist.*, vol. 19, no. 3, pp. 326–339, Sep. 1948.

[35] R. H. Berk, "Some asymptotic aspects of sequential analysis," *Ann. Statist.*, vol. 1, no. 6, pp. 1126–1138, Nov. 1973.

[36] M. I. Skolnik, *Introduction to Radar Systems*, 3rd ed. New York: McGraw-Hill, 2001.

[37] J. J. Bussgang and M. Marcus, "Truncated sequential hypothesis tests," *IEEE Trans. Inf. Theory*, vol. IT-13, no. 3, pp. 512–516, Jul. 1967.

[38] S. Tantaratana and H. V. Poor, "Asymptotic efficiencies of truncated sequential tests," *IEEE Trans. Inf. Theory*, vol. IT-28, no. 6, pp. 911–923, Nov. 1982.

[39] S. Tantaratana, "Design of nonparametric truncated sequential detectors with parallel linear boundaries," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 25, no. 4, pp. 483–490, July 1989.

[40] B. G. Leroux, "Maximum-likelihood estimation for hidden Markov models," *Stochastic Processes Appl.*, vol. 40, no. 1, pp. 127–143, Feb. 1992.

[41] J. F. C. Kingman, "Subadditive processes," in *Ecole d'Eté de Probabilités de Saint-Flour, V-1975 (Lecture Notes in Mathematics)*. Berlin/Heidelberg, Germany: Springer-Verlag, 1976, vol. 539, pp. 167–223.

[42] A. N. Shiryaev, *Probability*, 2nd ed. New York: Springer-Verlag, 1995.

[43] C. D. Fuh, "On Bahadur efficiency of the maximum likelihood estimator in hidden Markov models," *Statist. Sinica*, vol. 14, no. 1, pp. 127–154, Jan. 2004.

[44] P. Bougerol, "Théorèmes limite pour les systèmes linéaires à coefficients markoviens," *Probab. Theory Rel. Fields*, vol. 78, no. 2, pp. 193–221, June 1988.

[45] O. Cappé, E. Moulines, and T. Rydén, *Inference in Hidden Markov Models*. Berlin, Germany: Springer-Verlag, 2005.

[46] A. Dembo and O. Zeituni, *Large Deviations Techniques and Applications*, 2nd ed. New York: Springer-Verlag, 1998.

[47] R. S. Ellis, "Large deviations for a large class of random vectors," *Ann. Probab.*, vol. 12, no. 1, pp. 1–12, Feb. 1984.

[48] T. L. Lai, "On Chernoff-Savage statistics and sequential rank tests," *Ann. Statist.*, vol. 3, no. 4, pp. 825–845, Jul. 1975.

[49] T. L. Lai, "Convergence rates and r-quick versions of the strong law for stationary mixing sequences," *Ann. Probab.*, vol. 5, no. 5, pp. 693–706, Oct. 1977.